Analysis in Theory and Applications
Anal. Theory Appl., Vol. **37**, No. 4 (2021), pp. 520-540

DOI: 10.4208/ata.OA-2020-0040
December 2021

Regularity of Inhomogeneous Quasi-Linear Equations on the Heisenberg Group

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Received 9 September 2020; Accepted (in revised version) 3 October 2020

Abstract. We establish Hölder continuity of the horizontal gradient of weak solutions to quasi-linear p-Laplacian type non-homogeneous equations in the Heisenberg Group.

Key Words: Non-homogeneous.

AMS Subject Classifications: 35J62, 35J92, 35H20, 35R03

1 Introduction

The $C^{1,\alpha}$ regularity of the p-Laplacian has been established earlier in, for instance [6,15,19] in the Euclidean setting. Its sub-elliptic analogue for homgeneous sub-elliptic equations of p-Laplacian type on the Heisenberg group, was unavailable until [17, 21], in the last years. It is therefore natural to consider the case of regularity for the corresponding inhomogeneous equation and this is the purpose of the present contribution.

In this paper, we consider the equation

$$-\operatorname{div}_H a(x,\mathfrak{X}u)=\mu \quad \text{in } \Omega\subseteq\mathbb{H}^n, \tag{1.1}$$

where Ω is a domain and μ is a Radon measure with $|\mu|(\Omega)<\infty$ and $\mu(\mathbb{H}^n\setminus\Omega)=0$; hence Eq. (1.1) can be considered as defined in all of $\mathbb{H}^n$. Here we denote $\mathfrak{X}u=(X_1u,\cdots,X_{2n}u)$ as the horizontal gradient of $u:\Omega\to\mathbb{R}$, see Section 2.

We shall take up the following structural assumptions throughout the paper: the continuous function $a:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is assumed to be C^1 in the gradient variable and

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satisfies the following structure condition for every $x,y\in\Omega$ and $z,\xi\in\mathbb{R}^{2n}$,

$$(|z|^2+s^2)^{\frac{p-2}{2}}|\xi|^2\le\langle D_za(x,z)\xi,\xi\rangle\le L(|z|^2+s^2)^{\frac{p-2}{2}}|\xi|^2, \tag{1.2a}$$

$$|a(x,z)-a(y,z)|\le L'|z|(|z|^2+s^2)^{\frac{p-2}{2}}|x-y|^{\alpha}, \tag{1.2b}$$

where $L,L'\ge 1$, $s\ge 0$, $\alpha\in(0,1]$ and $D_za(x,z)$ is a symmetric matrix for every $x\in\Omega$. The sub-elliptic p-Laplacian equation with measure data, given by

$$-\operatorname{div}_H(|\mathfrak{X}u|^{p-2}\mathfrak{X}u)=\mu, \tag{1.3}$$

is a prototype of Eq. (1.1) with the condition (1.2) for the case $s=0$. The weak solutions of (1.1) are defined in horizontal Sobolev space $HW^{1,p}(\Omega)$; the Lipschitz and Hölder classes, denoted by same classical notations, are defined with respect to the CC-metric $(x,y)\mapsto d(x,y)$, see Section 2 for details. We shall denote $Q=2n+2$ as the homogeneous dimension. Now we state our main result.

Theorem 1.1. *Let $u\in HW^{1,p}(\Omega)$ be a weak solution of Eq.* (1.1) *with $p\ge 2$ and a C^1-function $a:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ satisfying the structure condition* (1.2). *If we have $\mu=f\in L^q_{\rm loc}(\Omega)$ for some $q>Q$, then $\mathfrak{X}u$ is locally Hölder continuous and there exists $c=c(n,p,L)>0$ and $\bar R=\bar R(n,p,L,L',\alpha,q,\operatorname{dist}(x_0,\partial\Omega))>0$ such that for any $x_0\in\Omega$, $0<R\le\bar R$ and $x,y\in B_R(x_0)\subset\Omega$, the estimate*

$$|\mathfrak{X}u(x)-\mathfrak{X}u(y)|\le c\,d(x,y)^{\gamma}\left(\fint_{B_R(x_0)}(|\mathfrak{X}u|+s)\,dx+\|f\|^{1/(p-1)}_{L^q(B_R(x_0))}\right), \tag{1.4}$$

holds for some $\gamma=\gamma(n,p,L,\alpha,q)\in(0,1)$. In particular, if $a(x,z)$ is independent of x, then (1.4) *holds for $\bar R=\bar R(n,p,L,\operatorname{dist}(x_0,\partial\Omega))>0$ and $\gamma(n,p,L,q)\in(0,1)$.*

The proof of Theorem 1.1 in this paper, relies on novel techniques introduced by Duzaar-Mingione [7] based on sharp comparison estimates of homogeneous equations with frozen coefficients, in other words, harmonic replacements. However, in the present sub-elliptic setting, one encounters extra terms coming from commutators of the horizontal vector fields which lead to estimates that are not always as strong as those in the Euclidean setting. An instance appears in Proposition 3.1 for the integral decay estimate, where the extra term in (3.1) appears unavoidably and can not be removed unlike similar integral estimates obtained previously in the Euclidean setting in [7,16], see Remark 3.1. Hence, one gets a weaker integral decay estimate of the oscillation of the gradient of solutions of the in-homogeneous solution. Nevertheless, a perturbation lemma (Lemma 4.2), similar to the standard lemma of Campanato [3,9], leads to the $C^{1,\alpha}$-regularity of weak solutions of Eq. (1.1) by exploiting the high integrability of the data.

We develop necessary notations, definitions and provide previous results on sub-elliptic equations in Section 2. Then we prove the intermediate estimates in Section 3 and finally, we prove Theorem 1.1 in Section 4.