Regularity Results for a Nonlinear Elliptic-Parabolic System with Oscillating Coefficients

Xiangsheng Xu

Department of Mathematics & Statistics, Mississippi State University, Mississippi State, MS 39762, USA

Received 9 May 2020; Accepted (in revised version) 4 January 2021

Abstract. In this paper we study the initial boundary value problem for the system
\[ \text{div}(\sigma(u) \nabla \varphi) = 0, \quad u_t - \Delta u = \sigma(u)|\nabla \varphi|^2. \]
This problem is known as the thermistor problem which models the electrical heating of conductors. Our assumptions on \( \sigma(u) \) leave open the possibility that \( \lim_{u \to \infty} \sigma(u) = 0 \), while \( \limsup_{u \to \infty} \sigma(u) \) is large. This means that \( \sigma(u) \) can oscillate wildly between 0 and a large positive number as \( u \to \infty \). Thus our degeneracy is fundamentally different from the one that is present in porous medium type of equations. We obtain a weak solution \((u, \varphi)\) with \( |\nabla \varphi|, |\nabla u| \in L^\infty \) by first establishing a uniform upper bound for \( \varepsilon u \) for some small \( \varepsilon \). This leads to an inequality in \( \nabla \varphi \), from which the regularity result follows. This approach enables us to avoid first proving the Hölder continuity of \( \varphi \) in the space variables, which would have required that the elliptic coefficient \( \sigma(u) \) be an \( A_2 \) weight. As it is known, the latter implies that \( \ln \sigma(u) \) is “nearly bounded”.

Key Words: Oscillating coefficients, the thermistor problem, quadratic nonlinearity.

AMS Subject Classifications: 35B45, 35B65, 35M33, 35Q92

1 Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with sufficiently smooth boundary \( \partial \Omega \) and \( T \) any positive number. We consider the initial boundary value problem

\[ u_t - \Delta u = \sigma(u)|\nabla \varphi|^2 \quad \text{in} \quad \Omega_T, \tag{1.1a} \]

\[ \text{div}(\sigma(u) \nabla \varphi) = 0 \quad \text{in} \quad \Omega_T, \tag{1.1b} \]

\[ u = u_0 \quad \text{on} \quad \partial \Omega_T, \tag{1.1c} \]

\[ \varphi = \varphi_0 \quad \text{on} \quad \Sigma_T. \tag{1.1d} \]

*Corresponding author. Email addresses: xxu@math.msstate.edu (X. Xu)
where
\[ \Omega_T = \Omega \times (0, T), \quad (1.2a) \]
\[ \Sigma_T = \partial \Omega \times (0, T), \quad \text{the lateral boundary of } \Omega_T, \quad (1.2b) \]
\[ \partial_t \Omega_T = \Sigma_T \cup \Omega \times \{0\}, \quad \text{the parabolic boundary of } \Omega_T. \quad (1.2c) \]

We are interested in the regularity properties of weak solutions when the elliptic coefficient \( \sigma(u) \) in the second equation may become oscillatory as \( u \to \infty \). To be precise, we establish the following

**Theorem 1.1 (Main Theorem).** Assume:

(H1) the function \( \sigma \) is continuously differentiable on the interval \([0, \infty)\) with
\[ c_0 e^{-\beta s} \leq \sigma(s) \leq c_1 \quad \text{on } [0, \infty) \quad \text{for some } c_0, c_1, \beta \in (0, \infty), \quad (1.3a) \]
\[ |\sigma'(s)| \leq c_2 e^{\gamma s} \quad \text{on } [0, \infty) \quad \text{for some } c_2, \gamma \in (0, \infty), \quad (1.3b) \]

(H2) \( u_0, \varphi_0 \in C([0, T]; C^1(\Omega)) \) with \( u_0|_{\partial_t \Omega_T} \geq 0 \) and \( \partial_t u_0 - \Delta u_0 \in L^s(\Omega_T), \Delta \varphi_0 \in L^\infty(0, T; L^s(\Omega)) \) for each \( s > 1 \),

(H3) \( \partial \Omega \) is \( C^{1,1} \).

Then there is a unique weak solution \((u, \varphi)\) to (1.1a)-(1.1d) with \( u \geq 0 \) and
\[ \nabla u, \nabla \varphi \in L^\infty(\Omega_T). \quad (1.4) \]

The notion of a weak solution is defined as follows:

**Definition 1.1.** We say that \((u, \varphi)\) is a weak solution to (1.1a)-(1.1d) if

(D1) \( u, \varphi \in L^2(0, T; W^{1,2}(\Omega)) \),

(D2) \( u = u_0, \varphi = \varphi_0 \) on \( \Sigma_T \) in the sense of the trace theorem and
\[ -\int_{\Omega_T} u \xi_t dx dt + \int_{\Omega_T} \nabla u \nabla \xi dx dt \]
\[ = \int_{\Omega_T} \sigma(u) |\nabla \varphi|^2 dx dt + \int_{\Omega_T} u_0(x, 0) \xi(x, 0) dx, \quad (1.5a) \]
\[ \int_{\Omega_T} \sigma(u) \nabla \varphi \nabla \eta dx dt = 0, \quad (1.5b) \]

for each pair of smooth functions \( \xi, \eta \) with \( \xi = \eta = 0 \) on \( \Sigma_T \) and \( \xi(x, T) = \eta(x, T) = 0 \).