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# Regularity Results for a Nonlinear Elliptic-Parabolic System with Oscillating Coefficients

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**Abstract.** In this paper we study the initial boundary value problem for the system $\operatorname{div}(\sigma(u)\nabla\varphi) = 0$, $u_t - \Delta u = \sigma(u)|\nabla\varphi|^2$. This problem is known as the thermistor problem which models the electrical heating of conductors. Our assumptions on $\sigma(u)$ leave open the possibility that $\liminf_{u\to\infty}\sigma(u) = 0$, while $\limsup_{u\to\infty}\sigma(u)$ is large. This means that $\sigma(u)$ can oscillate wildly between 0 and a large positive number as $u \to \infty$. Thus our degeneracy is fundamentally different from the one that is present in porous medium type of equations. We obtain a weak solution $(u,\varphi)$ with $|\nabla\varphi|, |\nabla u| \in L^\infty$ by first establishing a uniform upper bound for $e^{\varepsilon u}$ for some small $\varepsilon$. This leads to an inequality in $\nabla\varphi$, from which the regularity result follows. This approach enables us to avoid first proving the Hölder continuity of $\varphi$ in the space variables, which would have required that the elliptic coefficient $\sigma(u)$ be an $A_2$ weight. As it is known, the latter implies that $\ln\sigma(u)$ is "nearly bounded".

**Key Words**: Oscillating coefficients, the thermistor problem, quadratic nonlinearity.

**AMS Subject Classifications**: 35B45, 35B65, 35M33, 35Q92

## 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and $T$ any positive number. We consider the initial boundary value problem

$$u_t - \Delta u = \sigma(u)|\nabla\varphi|^2 \qquad \text{in } \Omega_T, \tag{1.1a}$$
$$\operatorname{div}(\sigma(u)\nabla\varphi) = 0 \qquad \text{in } \Omega_T, \tag{1.1b}$$
$$u = u_0 \qquad \text{on } \partial_p\Omega_T, \tag{1.1c}$$
$$\varphi = \varphi_0 \qquad \text{on } \Sigma_T, \tag{1.1d}$$

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where

$$\Omega_T = \Omega \times (0,T), \tag{1.2a}$$
$$\Sigma_T = \partial\Omega \times (0,T), \qquad \text{the lateral boundary of } \Omega_T, \tag{1.2b}$$
$$\partial_p\Omega_T = \Sigma_T \cup \Omega \times \{0\}, \qquad \text{the parabolic boundary of } \Omega_T. \tag{1.2c}$$

We are interested in the regularity properties of weak solutions when the elliptic coefficient $\sigma(u)$ in the second equation may become oscillatory as $u \to \infty$. To be precise, we establish the following

**Theorem 1.1** (Main Theorem)**.** *Assume:*

*(H1) the function $\sigma$ is continuously differentiable on the interval $[0,\infty)$ with*

$$c_0 e^{-\beta s} \le \sigma(s) \le c_1 \qquad \text{on } [0,\infty) \text{ for some } c_0, c_1, \beta \in (0,\infty), \tag{1.3a}$$
$$|\sigma'(s)| \le c_2 e^{\gamma s} \qquad \text{on } [0,\infty) \text{ for some } c_2, \gamma \in (0,\infty), \tag{1.3b}$$

*(H2) $u_0, \varphi_0 \in C\left([0,T]; C^1(\overline{\Omega})\right)$ with $u_0|_{\partial_p\Omega_T} \ge 0$ and $\partial_t u_0 - \Delta u_0 \in L^s(\Omega_T)$, $\Delta\varphi_0 \in L^\infty(0,T;L^s(\Omega))$ for each $s > 1$,*

*(H3) $\partial\Omega$ is $C^{1,1}$.*

*Then there is a unique weak solution $(u,\varphi)$ to* (1.1a)-(1.1d) *with $u \ge 0$ and*

$$\nabla u, \nabla\varphi \in L^\infty(\Omega_T). \tag{1.4}$$

The notion of a weak solution is defined as follows:

**Definition 1.1.** *We say that $(u,\varphi)$ is a weak solution to* (1.1a)-(1.1d) *if*

*(D1) $u, \varphi \in L^2(0,T;W^{1,2}(\Omega))$,*

*(D2) $u = u_0, \varphi = \varphi_0$ on $\Sigma_T$ in the sense of the trace theorem and*

$$-\int_{\Omega_T} u\xi_t dxdt + \int_{\Omega_T} \nabla u \nabla\xi dxdt = \int_{\Omega_T} \sigma(u)|\nabla\varphi|^2 dxdt + \int_\Omega u_0(x,0)\xi(x,0)dx, \tag{1.5a}$$
$$\int_{\Omega_T} \sigma(u)\nabla\varphi\nabla\eta dxdt = 0, \tag{1.5b}$$

*for each pair of smooth functions $\xi,\eta$ with $\xi = \eta = 0$ on $\Sigma_T$ and $\xi(x,T) = \eta(x,T) = 0$.*