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On the Projections of the Mutual Multifractal Rényi Dimensions

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Abstract. In this paper, we compare the mutual multifractal Rényi dimensions to the mutual multifractal Hausdorff and pre-packing dimensions. We also provide a relationship between the mutual multifractal Rényi dimensions of orthogonal projections of a couple of measures (μ,ν) in $\mathbb{R}^n$. As an application, we study the mutual multifractal analysis of the projections of measures.

Key Words: Mutual Hausdorff dimension, mutual packing dimension, mutual multifractal analysis, doubling measures, projection.

AMS Subject Classifications: 28A20, 28A80

1 Introduction and statement of the results

Singularity, exponents or spectrum and generalized dimensions are the major components of the multifractal analysis. Recently, the projection behavior of dimensions and multifractal spectra of measures have generated a large interest in the mathematical literature [5, 6, 11, 25, 28, 29, 31–33, 35–38, 43, 44, 50, 52–55]. The study of the behavior of Hausdorff dimension under projection type mappings dates back to the 50's when Marstrand [42] proved a well-known theorem according to which the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. Kaufman [38] proved the same result using potential theoretic methods. Also Mattila's proof [43] for the general case is based on the potential theoretic approach that was later generalized to higher dimensions by Hu and Taylor [36] and for the Hausdorff dimension of a measure by Falconer and Mattila [29].

The behavior of the packing dimension under projections is not as straightforward as that of the Hausdorff dimension. While the Hausdorff dimension of a set or a measure is preserved under almost all projections, the packing dimension may decrease for almost

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all of them. However, in [28] Falconer and Howroyd proved that the packing dimension of the projected set or measure will be the same for almost all projections.

As a continuity to these research many authors have studied the relationship between multifractal features of a measure μ on $\mathbb{R}^n$ and those of the projection of the measure onto m-dimensional subspaces [6, 25, 26, 50, 54]. Other works were carried in this sense for classes of similar measures in Euclidean and symbolic spaces [35, 52, 53].

Recently, mixed multifractal spectra have generated an enormous interest in the mathematical literature. Many authors were interested in mixed multifractal spectra and their applications [1, 7–10, 15, 16, 18–23, 33, 34, 45, 46, 59]. Previously, only the scaling behavior

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}$$

of a single measure μ has been investigated (see for example [12–14, 31, 37, 40, 47, 49, 50]). However, mixed multifractal analysis of two Borel probability measures μ_1 and μ_2 (mutual multifractal analysis) on $\mathbb{R}^n$ investigates the simultaneous scaling behavior

$$\lim_{r\to 0}\frac{\log\mu_1(B(x,r))}{\log r},\quad \lim_{r\to 0}\frac{\log\mu_2(B(x,r))}{\log r}.$$

It combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system and provides the basis for a significantly better understanding of the underlying dynamics. Olsen [48] conjectured a mixed multifractal formalism which links the mixed spectrum to the Legendre transform of mixed Rényi dimensions. Olsen obtained a general upper bound. He also proved that this bound is equality if both measures are self-similar with same contracting similarities. Later, in [45, 46], a mixed multifractal formalism associated with the mixed multifractal generalizations of Hausdorff and packing measures and dimensions is proved in some cases based on a generalization of the well known large deviation formalism.

In the present paper, we pursue those kinds of studies and we consider the mixed multifractal analysis of two Borel probability measures developed in [45,58,60–63]. Firstly, we compare the mutual multifractal Rényi dimensions to the mutual multifractal Hausdorff and pre-packing dimensions. Secondly, we investigate a relationship between the mutual multifractal Rényi dimensions and its projection onto a lower dimensional linear subspace. As an application, we study the mutual multifractal analysis of the projections of measures.

2 Preliminaries

In this section, we aim to introduce the general tools that will be applied next. We will review in brief the notion of mutual multifractal Hausdorff and packing measures already introduced in [45, 46, 61, 64].