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Nodal Solutions of the Brezis-Nirenberg Problem in Dimension 6

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Abstract. We show that the classical Brezis-Nirenberg problem

$$\begin{aligned} -\Delta u &= u|u| + \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

when Ω is a bounded domain in $\mathbb{R}^6$ has a sign-changing solution which blows-up at a point in Ω as λ approaches a suitable value $\lambda_0 > 0$.

Key Words: Sign-changing solutions, blow-up phenomenon, Ljapunov-Schmidt reduction, Transversality theorem.

AMS Subject Classifications: 35B44, 58C15

1 Introduction

Brezis and Nirenberg in their famous paper [6] introduced the problem

$$-\Delta u = |u|^{\frac{4}{n-2}} u + \lambda u \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $n \geq 3$. A huge number of results concerning (1.1) has been obtained since then. Let us summarize the most relevant results which are also connected with the topic of the present paper.

First of all, the classical Pohozaev's identity ensures that (1.1) does not have any solutions if $\lambda \leq 0$ and Ω is a star-shaped domain. A simple argument shows that problem

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(1.1) does not have any positive solutions if $\lambda \geq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. The existence of a least energy positive solution u_λ to (1.1), i.e., a solution which achieves the infimum

$$m_\lambda := \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\int_{\Omega_\theta}\left(|\nabla u|^2-\lambda u^2\right)dx}{\left(\int_\Omega |u|^{p+1}dx\right)^{\frac{2}{p+1}}}$$

has been proved by Brezis and Nirenberg in [6] when $\lambda \in (0,\lambda_1(\Omega))$ in dimension $n \geq 4$ and when $\lambda \in (\lambda^*(\Omega),\lambda_1(\Omega))$ in dimension $n=3$, where $\lambda^*(\Omega)>0$ depends on the domain Ω. If Ω is the ball then $\lambda^*(\Omega)=\frac{1}{4}\lambda_1(\Omega)$ (see [6]), while the general case has been treated by Druet in [12]. The existence of a sign-changing solution has been proved by Cerami, Solimini and Struwe in [9] when $\lambda \in (0,\lambda_1(\Omega))$ and $n \geq 6$ and by Capozzi, Fortunato and Palmieri in [8] when $\lambda \geq \lambda_1(\Omega)$ and $n \geq 4$.

There is a wide literature about the study of the asymptotic profile of the solutions when the parameter λ approaches either zero or some strictly positive values depending on the dimension n and the domain Ω. In the following, we will focus on the existence of solutions which exhibit a positive or negative blow-up phenomenon as λ approaches some particular values.

When the parameter λ approaches zero, positive and sign-changing solutions which blow-up positively or negatively at one or more points in Ω do exist provided the dimension $n \geq 4$. Rey in [24], Musso and Pistoia in [19] and Esposito, Pistoia and Vétois in [13] built solutions to (1.1) with simple positive or negative blow-up points, i.e., around each point the solution looks like a positive or a negative standard bubble. Here the standard bubbles are the functions

$$U_{\delta,\xi}(x) := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{\left(\delta^2+|x-\xi|^2\right)^{\frac{n-2}{2}}} \quad \text{with } \delta>0, \quad \xi \in \mathbb{R}^n, \tag{1.2a}$$

$$\alpha_n := \left(n(n-2)\right)^{\frac{n-2}{4}}, \tag{1.2b}$$

which are the only positive solutions of the equation

$$-\Delta U = U^{\frac{n+2}{n-2}}$$

in $\mathbb{R}^n$ (see [4,7,28]) More precisely, if λ is small enough problem (1.1) has a positive solution which blows-up at one point (see [24] if $n \geq 5$ and [13] if $n=4$) and a sign-changing solution which blows-up positively and negatively at two different points (see [19] if $n \geq 5$). As far as we know the existence of multiple concentration in the case $n=4$ is still open. If $n=3$ positive solutions of (1.1) blowing-up at a single point when the parameter λ approaches a strictly positive number have been found by Del Pino, Dolbeault and Musso in [11]. Moreover, sign-changing solutions having both positive and negative blow-up points can be constructed arguing as Musso and Salazar in [20], where they found solutions which blow-up at more points when λ is close to a suitable strictly positive number. In higher dimension $n \geq 7$ Premoselli [22] found an arbitrary large number