Analysis in Theory and Applications
Anal. Theory Appl., Vol. **38**, No. 1 (2022), pp. 110-120

DOI: 10.4208/ata.OA-2017-0063
March 2022

On Proximal Relations in Transformation Semigroups Arising from Generalized Shifts

Fatemah Ayatollah Zadeh Shirazi[1,*], Amir Fallahpour[2], Mohammad Reza Mardanbeigi[3] and Zahra Nili Ahmadabadi[3]

[1] *Faculty of Mathematics Statistics and Computer Science, College of Science, University of Tehran, Enghelab Ave., Tehran, Iran*
[2] *Faculty of Mathematics and Computer Science, College of Science, Amirkabir University of Technology, Tehran, Iran*
[3] *Islamic Azad University, Science and Research Branch Tehran, Iran*

Received 16 October 2017; Accepted (in revised version) 24 September 2019

Abstract. For a finite discrete topological space X with at least two elements, a nonempty set Γ, and a map $\varphi:\Gamma\to\Gamma$, $\sigma_\varphi: X^\Gamma\to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma})=(x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$) is a generalized shift. In this text for $\mathcal{S}=\{\sigma_\psi:\psi\in\Gamma^\Gamma\}$ and $\mathcal{H}=\{\sigma_\psi:\Gamma\overset{\psi}{\to}\Gamma \text{ is bijective}\}$ we study proximal relations of transformation semigroups $(\mathcal{S},X^\Gamma)$ and $(\mathcal{H},X^\Gamma)$. Regarding proximal relation we prove:

$$P(\mathcal{S},X^\Gamma)=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in X^\Gamma\times X^\Gamma:\exists\beta\in\Gamma\ (x_\beta=y_\beta)\}$$

and $P(\mathcal{H},X^\Gamma)\subseteq\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in X^\Gamma\times X^\Gamma:\{\beta\in\Gamma:x_\beta=y_\beta\}\text{ is infinite}\}\cup\{(x,x):x\in\mathcal{X}\}$.

Moreover, for infinite Γ, both transformation semigroups $(\mathcal{S},X^\Gamma)$ and $(\mathcal{H},X^\Gamma)$ are regionally proximal, i.e., $Q(\mathcal{S},X^\Gamma)=Q(\mathcal{H},X^\Gamma)=X^\Gamma\times X^\Gamma$, also for sydetically proximal relation we have $L(\mathcal{H},X^\Gamma)=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in X^\Gamma\times X^\Gamma:\{\gamma\in\Gamma:x_\gamma\neq y_\gamma\}\text{ is finite}\}$.

Key Words: Generalized shift, proximal relation, transformation semigroup.

AMS Subject Classifications: 54H15, 37B09

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1 Preliminaries

By a (left topological) transformation semigroup (S,Z,π) or simply (S,Z) we mean a compact Hausdorff topological space Z (phase space), discrete topological semigroup S

*Corresponding author. *Email addresses:* `fatemah@khayam.ut.ac.ir`, `f.a.z.shirazi@ut.ac.ir` (F. Ayatollah Zadeh Shirazi), `amir.falah90@yahoo.com` (A. Fallahpour), `mrmardanbeigi@srbiau.ac.ir` (M. Mardanbeigi), `zahra.nili.a@gmail.com` (Z. Nili Ahmadabadi)

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(phase semigroup) with identity e and continuous map $\pi : S \times Z \to Z$ ($\pi(s,z) = sz, s \in S, z \in Z$) such that for all $z \in Z$ and $s,t \in S$ we have $ez = z$, $(st)z = s(tz)$. If S is a discrete topological group too, then we call the transformation semigroup (S,Z), a *transformation group*. We say $(x,y) \in Z \times Z$ is a *proximal pair* of (S,Z) if there exists a net $\{s_\lambda\}_{\lambda\in\Lambda}$ in S with

$$\lim_{\lambda\in\Lambda} s_\lambda x = \lim_{\lambda\in\Lambda} s_\lambda y.$$

We denote the collection of all proximal pairs of (S,Z) by $P(S,Z)$ and call it *proximal relation* on (S,Z), for more details on proximal relations we refer the interested reader to [4,8].

In the transformation semigroup (S,Z) we call $(x,y) \in Z \times Z$ a regionally proximal pair if there exists a net $\{(s_\lambda, x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ in $S \times Z \times Z$ such that

$$\lim_{\lambda\in\Lambda} x_\lambda = x, \quad \lim_{\lambda\in\Lambda} y_\lambda = y \quad \text{and} \quad \lim_{\lambda\in\Lambda} s_\lambda x_\lambda = \lim_{\lambda\in\Lambda} s_\lambda y_\lambda.$$

We denote the collection of all regionally proximal pairs of (S,Z) by $Q(S,Z)$ and call it regionally proximal relation on (S,Z). Obviously we have $P(S,Z) \subseteq Q(S,Z)$. In the transformation group (T,Z), by [9] we call $L(T,Z) = \{(x,y) \in Z \times Z : \overline{T(x,y)} \subseteq P(T,Z)\}$ the syndetically proximal relation of (T,Z) (for details on the interaction of $L(T,Z)$, $Q(T,Z)$ and $P(T,Z)$ with uniform structure of Z see [5,6,9]).

1.1 A collection of generalized shifts as phase semigroup

For nonempty sets X, Γ and self-map $\varphi : \Gamma \to \Gamma$ define the generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ by $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ $((x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma)$. Generalized shifts have been introduced for the first time in [2], in addition dynamical and non-dynamical properties of generalized shifts have been studied in several texts like [3] and [7]. It's well-known that if X has a topological structure, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous (when X^Γ equipped with product topology), in addition If X has at least two elements, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is a homeomorphism if and only if $\varphi : \Gamma \to \Gamma$ is bijective.

Convention. In this text suppose X is a finite discrete topological space with at least two elements, Γ is a nonempty set, $\mathcal{X} := X^\Gamma$, and:

- $\mathcal{S} := \{\sigma_\varphi : \varphi \in \Gamma^\Gamma\}$, is the semigroup of generalized shifts on X^Γ,
- $\mathcal{H} := \{\sigma_\varphi : \varphi \in \Gamma^\Gamma$ and $\varphi : \Gamma \to \Gamma$ is bijective$\}$, is the group of generalized shift homeomorphisms on X^Γ.

Equip X^Γ with product (pointwise convergence) topology. Now we may consider $\mathcal{S}$ (resp. $\mathcal{H}$) as a subsemigroup (resp. subgroup) of continuous maps (resp. homeomorphisms) from $\mathcal{X}$ to itself, so $\mathcal{S}$ (resp. $\mathcal{H}$) acts on $\mathcal{X}$ in a natural way.

Our aim in this text is to study $P(T,\mathcal{X})$, $Q(T,\mathcal{X})$, and $L(T,\mathcal{X})$ for $T = \mathcal{H}, \mathcal{S}$. Readers interested in this subject may refer to [1] too.