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Singular Solutions to Monge-Ampère Equation

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Abstract. We construct merely Lipschitz and $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ viscosity solutions to the Monge-Ampère equation with constant right hand side.

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1 Introduction

In this note, we construct (convex) Lipschitz and $C^{1,\alpha}$ viscosity solutions to the Monge-Ampère equation with constant right hand side via Cauchy-Kovalevskaya, after integerizing fractional powers in the corresponding equation for those singular profiles from [8] and [3,5].

Theorem 1.1. *There exists a merely Lipschitz viscosity solution to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$. *There also exist merely* $C^{1,\alpha-1}$ *with rational* $\alpha = \frac{q}{p} \in (1, 2-\frac{2}{n}]$ *viscosity solutions to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$.

These $C^{1,\alpha}$ solutions to the Monge-Ampère equation $\det D^2u = 1$ illustrate a regularity wall phenomena: merely $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ solutions can never become better. This is in contrast with the regularity theory for Monge-Ampère equations [9] and [4]: once solutions are $C^{1,(1-2/n)+}$, they self-improve to smoothness.

Note that our singular solutions via Cauchy-Kovalevskaya to the Monge-Ampère equation $\det D^2u = 1$ are singular precisely along a segment of one axis, where the convex solutions are linear, or zero, to be precise. If one tries to produce higher dimensional

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subspace singular set, where the dimension S must be less than $n/2$ by the theorem in [3], a good start is the Pogorelov type profile there,

$$u(x) = |x'|^{2-2S/n} f(|x''|)$$

with $x = \left(x'_1, \cdots, x'_{n-S}, x''_1, \cdots, x''_S\right)$. The profile with

$$f(|x''|) = 1 + |x''|^2$$

satisfies the Monge-Ampère with the right hand side being a polynomial of $|x''|^2$, positive near the origin. The ODE for $f(|x''|)$ with singular term $f'(|x''|)/|x''|$ corresponding to $\det D^2u = 1$ can be solved by the method in [2] and [1].

Alternatively, relying on the existence of solutions to the Dirichlet problem for Monge-Ampère equations, with S dimensional singular set profile $|x'|^{2-2S/n}(1+|x''|^2)$ as boundary value in a small ball, one obtains the following

Proposition 1.1. *There exist local merely $C^{1,1-S/n}$ viscosity solutions to* $\det D^2u = 1$ *in* $\mathbb{R}^n$ *for* $n \geq 3$ *such that singular set of the solutions is the S dimensional set*

$$\mathbb{S} = \left\{(x', x'') : |x'| = 0\right\}$$

in a small ball for $1 \leq S < n/2$.

Let us sketch a proof for this proposition. Case $S = 1$ is also noted in the above. The Lipschitz limit of a family of (convex) smooth solutions to $\det D^2u = 1$ with smooth boundary value approximations of subsolution

$$u_- = \gamma |x'|^{2-2S/n}\left(1 + |x''|^2\right)$$

for $\gamma = (1-2S/n)^{-1/n}$ on the boundary of a small ball is our viscosity solution. The convex solution $u(x)$ vanishes in subspace x'' with $|x'| = 0$, because it is between the convex combinations of zero boundary value and the subsolution u_- there. Surely $u(x)$ is singular in the S dimensional subspace $(0', x'')$.

We show that u is regular everywhere else. By [4,5], the other possible singular set of u outside $\mathbb{S}$, must contain a line segment, where u is linear. This singular segment intersects the boundary of the small ball or the set $\mathbb{S}$. The barrier argument in [9] and [4,5] shows the two ends of the segment cannot be both on the boundary of the small ball, where u is smooth. The only other scenario that the segment has one end on $\mathbb{S}$, and the other end on the boundary of the small ball is not possible either. This is because the linear function, the restriction of u on the segment, equaling 0 and $u_- > 0$ respectively on the two ends, cannot be less than the supersolution

$$u^+ = 2\gamma |x'|^{2-2S/n}$$