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Existence and Multiplicity Results for a Class of Nonlinear Schrödinger Equations with Magnetic Potential Involving Sign-Changing Nonlinearity

Francisco Odair de Paiva[1], Sandra Machado de Souza Lima[2] and Olimpio Hiroshi Miyagaki[1,*]

[1] *Departamento de Matemática, UFSCar, São Carlos, SP, 13560-970 Brazil*
[2] *Departamento de Ciências Exatas, Biológicas e da Terra, INFES-UFF, Santo Antônio de Pádua–RJ, 28470-000, Brazil*

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Abstract. In this work we consider the following class of elliptic problems

$$\begin{cases} -\Delta_A u + u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1_A(\mathbb{R}^N), \end{cases} \tag{P}$$

with $2 < q < p < 2^* = \frac{2N}{N-2}$, $a(x)$ and $b(x)$ are functions that can change sign and satisfy some additional conditions; $u \in H^1_A(\mathbb{R}^N)$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential. Also using the Nehari method in combination with other complementary arguments, we discuss the existence of infinitely many solutions to the problem in question, varying the assumptions about the weight functions.

Key Words: Magnetic potential, sign-changing weight functions, Nehari manifold, Fibering map.
AMS Subject Classifications: 35Q60, 35Q55,35B38, 35B33

1 Introduction

We are interested in studying the following class of elliptic problems

$$\begin{cases} -\Delta_A u + u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1_A(\mathbb{R}^N), \end{cases} \tag{P}$$

with $2 < q < p < 2^* = \frac{2N}{N-2}$, $a(x)$ and $b(x)$ are functions that can change sign and satisfy some additional conditions, $u \in H^1_A(\mathbb{R}^N)$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential. We

*Corresponding author. *Email addresses:* `odair@dm.ufscar.br` (F. Paiva), `sandra.msouzalima@gmail.com` (S. Lima), `ohmiyagaki@gmail.com` (O. Miyagaki)

will discuss the existence of infinitely many solutions to the problem in question, varying the hypotheses about the weight functions.

We will make use of the magnetic operator in which we work with the Magnetic Laplacian. Denote $\nabla_A u = (\nabla + iA)u$ and then we have

$$-\Delta_A u := (-i\nabla + A)^2 u.$$

The Magnetic Laplacian is given by $(-i\nabla + A)^2 + V$, where $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential and $V : \mathbb{R}^N \to \mathbb{R}$ is the electrical potential. We describe this operator and the $H^1_A(\mathbb{R}^N)$ space with more details later on. Its importance in physics was discussed in Alves and Figueiredo [3] and also in Arioli and Szulkin [5].

Still seeking to contextualize the problems that we deal with in this work, we will speak a little of what has been done with respect to problems of the convex type with the usual Laplacian. Starting with the work of Alama and Tarantello [2] who considered the problem

$$\begin{cases} -\Delta u - \lambda u = W(x) f(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

In this case W is a function that can change sign and f satisfies

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|^{p-2}u} = a > 0$$

for some $2 < p \leq 2^*$ ($2 < p < \infty$ if $N = 1$ or 2). They deal with the existence of positive solutions. In the case where f is an odd function, they show the existence of infinitely many solutions which may even be solutions that change sign.

In [7], Berestycki et al. studied the existence and non-existence of solutions to the following class of problems

$$\begin{cases} -\Delta u + m(x)u = a(x)u^p, & x \in \Omega, \\ Bu(x) = 0, & x \in \partial\Omega. \end{cases}$$

In this case, $m(x)$ can change sign, $1 < p < 2^* - 1$ and $Bu = u$, $Bu = \partial_\nu u$ or $Bu = \partial_\nu u + \alpha(x)u$, $\alpha > 0$.

Other works that also dealt with the convex case in the bounded domain with the usual Laplacian can be seen in [1, 8, 9, 16, 25].

Already in $\mathbb{R}^N$, Miyagaki [19] studies the existence of nontrivial solutions for the following class of elliptic problems

$$-\Delta u + a(x)u = \lambda |u|^{q-1}u + |u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where $1 < p < q \leq 2^* - 1 = \frac{N+2}{N-2}$, $\lambda > 0$ is a constant and $a(x) : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying some additional conditions.