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Rearrangement Free Method for Hardy-Littlewood-Sobolev Inequalities on $\mathbb{S}^n$

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Abstract. For conformal Hardy-Littlewood-Sobolev(HLS) inequalities [22] and reversed conformal HLS inequalities [8] on $\mathbb{S}^n$, a new proof is given for the attainability of their sharp constants. Classical methods used in [22] and [8] depends on rearrangement inequalities. Here, we use the subcritical approach to construct the extremal sequence and circumvent the blow-up phenomenon by renormalization method. The merit of the method is that it does not rely on rearrangement inequalities.

Key Words: Hardy-Littlewood-Sobolev inequality, reversed Hardy-Littlewood-Sobolev inequality, rearrangement free method.

AMS Subject Classifications: 39B62, 26A33, 26D10

1 Introduction

The conformal Hardy-Littlewood-Sobolev(HLS) inequality on $\mathbb{R}^n$ is

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy\right| \le N_{n,\alpha}\|f\|_{p_\alpha}\|g\|_{p_\alpha}, \quad 0<\alpha<n, \quad p_\alpha=\frac{2n}{n+\alpha}, \tag{1.1}$$

where

$$N_{n,\alpha}=\pi^{(n-\alpha)/2}\frac{\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\alpha/n}$$

is the best constant. Lieb [22] proved that the extremal functions of (1.1) are radial symmetric by rearrangement inequalities, and obtained the sharp constant by the conformal symmetries of (1.1). Different discussions can be found in [3,23]. Recently, the classification of the solutions for the Euler-Lagrange equation of (1.1) was given in [4] and [21] by the method of moving planes and the method of moving spheres, respectively.

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For $\alpha > n$, Dou and Zhu [8] (also see [2,24]) established a class of reversed conformal HLS inequalities

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy \geq \tilde{N}_{n,\alpha}\|f\|_{p_\alpha}\|g\|_{p_\alpha}, \tag{1.2}$$

where $\tilde{N}_{n,\alpha} = N_{n,\alpha}$ is the best constant. Employing the rearrangement inequalities and the method of moving spheres, they obtained the sharp constant and classified the solutions of the corresponding Euler-Lagrange equation.

As stated above, it can be found that rearrangement inequalities, the method of moving planes and the method of moving spheres are basic and important tools for studying the HLS inequalities. More applications of these techniques can be found in the study of HLS type inequalities and reversed HLS type inequalities on the upper half space (see [6,9,18,25] and the references therein).

Heisenberg group is one of the simplest noncommutative geometries and is the model space of CR manifolds. It is natural that we want to generalize these traditional methods on Heisenberg group. But, because of the non-commutativity, rearrangement inequalities, the method of moving planes and the method of moving spheres don't work efficiently on Heisenberg group. In this paper, we will try a class of rearrangement free method and give a new proof for the existence of the extremal functions of (1.1) and (1.2). Recently, we successfully generalize the method to study the reversed HLS inequalities on the Heisenberg group (see [15]).

From [22] and [8], we know that the extremal functions of (1.1) and (1.2) have the form

$$f_\epsilon(x) = c_1 g_\epsilon(x) = c\left(\frac{\epsilon}{\epsilon^2+|x-x_0|^2}\right)^{(n+\alpha)/2},$$

where c_1, c and ϵ are constants, x_0 is some point in $\mathbb{R}^n$. Note that f_ϵ and g_ϵ will blow up as $\epsilon \to 0^+$, and vanish as $\epsilon \to +\infty$. The phenomenon makes it difficult to study the extremal problems. To overcome the difficulty, we often renormalize the extremal sequence. For example, Lieb [22] renormalized the extremal sequence $\{f_j(x)\}$ so that it satisfies $f_j(x) > \beta > 0$ if $|x| = 1$. The technique can also be found in [8].

Recently, Dou, Guo and Zhu [6] adopted a subcritical approach to study sharp HLS type inequalities on the upper half space. By Young inequality, they first established two classes of HLS type inequalities with subcritical power on a ball. Then, using the conformal transformation between ball and upper half space and the method of moving planes, they proved that the extremal functions of HLS type inequalities with subcritical power are constant functions. Passing to the limit from subcritical power to critical power, they obtained two classes of sharp HLS type inequalities on the upper half space.

In their approach, we note three advantages. First, extremal functions of HLS type inequalities with subcritical power satisfy the corresponding Euler-Lagrange equation, by which we can study the regularity of theses functions. Second, as power approach to critical, the corresponding extremal functions form a extremal sequence to the extremal problem of HLS type inequalities with critical power. Third, since these extremal functions of