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Fixed Point Theorems for Weakly Contractive Mappings in Ordered Metric Spaces with an Application

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Abstract. In this paper, we prove fixed point theorem for weakly contractive mappings using locally T-transitivity of binary relation and presenting an analogous version of Harjani and Sadarangani theorem involving more general relation theoretic metrical notions. Our fixed point results under universal relation reduces to Harjani and Sadarangani [Nonlinear Anal., 71 (2009), 3403–3410] fixed point theorems. In this way we also generalize some of the recent fixed point theorems for weak contraction in the existing literature.

Key Words: $\mathcal{R}$-continuity, locally T-transitive binary relation, weakly contractive map.

AMS Subject Classifications: 47H10, 54H25

1 Introduction

In 1997, Alber and Guerre-Delabrere [5] introduced the concept of weak contrction in Hilbert spaces and proved the corresponding fixed result. Later Rhoades [17] showed that the result is also valid in complete metric spaces. Further results in this direction were obtained by Dutta and Choudhury in [9]. Results on weakly contractive mappings in ordered metric spaces, together with applications to differential equations, were given by Harjani and Sadarangani in [10]. On the other hand, extension of classical Banach contraction principle [6] to the field of (partially) ordered metric spaces can be traced back to Turinici [21,24] which was later undertaken by several researchers [1,4,8,11,12, 14,15,17,19,20,22,23].

In all these extensions, we must notify the one due to Alam and Imdad [2], where some relation theoretic analogues of standard metric notions (such as continuity and completeness) were used. Further, Alam and Imdad [3] extended the above setting by

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using T-transitivity of the ambient relation $\mathcal{R}$, and obtained an extension of the Boyd-Wong Fixed Point Theorem [7] to such spaces. It is our aim in this paper is to give an extension of these results to weakly contractive maps.

2 Preliminaries

Throughout this paper, $\mathcal{R}$ stands for a non-empty binary relation N_0 stands for the set of whole numbers (i.e., $N_0 = N \cup \{0\}$) and R for the set of all real numbers.

In this section, we present some basic definitions, propositions and relevant relation-theoretic variants of metrical notions such as completeness and continuity.

Definition 2.1 ([16]). *A binary relation on a non-empty set X is defined as a subset of $X \times X$, which will be denoted by $\mathcal{R}$. We say that x relates to y under $\mathcal{R}$ iff $(x,y) \in \mathcal{R}$.*

Definition 2.2. *Let R be a binary relation on a nonempty set X and $x,y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$. We denote it by $[x,y] \in \mathcal{R}$.*

Definition 2.3 ([16]). *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.*

(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = (x,y) \in X^2 : (y,x) \in \mathcal{R}$.

(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup R^{-1}$ (i.e., $\mathcal{R}^s =: R \cup \mathcal{R}^{-1}$. Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 2.1 ([2]). *For a binary relation $\mathcal{R}$ defined on a nonempty set $X, (x,y) \in \mathcal{R}^s \iff [x,y] \in \mathcal{R}$.*

Definition 2.4 ([2]). *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}\ \forall n \in N_0$.*

Definition 2.5 ([2]). *Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ on X is called T-closed if for any $x,y \in X, (x,y) \in \mathcal{R} \Rightarrow (Tx,Ty) \in \mathcal{R}$.*

Proposition 2.2 ([2]). *Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. If $\mathcal{R}$ is T-closed, then $\mathcal{R}^s$ is also T-closed.*

Proposition 2.3 ([2]). *Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. If $\mathcal{R}$ is T-closed, then, for all $n \in N_0$, $\mathcal{R}$ is also T^n-closed, where T^n denotes n th iterate of T.*

Definition 2.6 ([3]). *Let (X,d) be a metric space and $\mathcal{R}$ a binary relation on X. We say that (X,d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

Clearly, every complete metric space is $\mathcal{R}$-complete, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.