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# A Review of Results on Axially Symmetric Navier-Stokes Equations, with Addendum by X. Pan and Q. Zhang

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**Abstract.** In this paper, we give a brief survey of recent results on axially symmetric Navier-Stokes equations (ASNS) in the following categories: regularity criterion, Liouville property for ancient solutions, decay and vanishing of stationary solutions. Some discussions also touch on the full 3 dimensional equations. Two results, closing of the scaling gap for ASNS and vanishing of homogeneous D solutions in 3 dimensional slabs will be described in more detail.

In the addendum, two new results in the 3rd category will also be presented, which are generalizations of recently published results by the author and coauthors.

**Key Words**: Regularity, Liouville theorem, ancient solutions, D-solutions, axially symmetric Navier-Stokes equations.

**AMS Subject Classifications**: 35Q30, 76N10

## 1 Introduction

The Cauchy problem of Navier-Stokes equations (NS) describing the motion of viscous incompressible fluids in $\mathbb{R}^3$ is

$$\begin{cases} \mu\Delta v - (v\cdot\nabla)v - \nabla P - \partial_t v = 0 \quad \text{on } \mathbb{R}^3\times(0,\infty), \\ div\, v = 0, \quad v(x,0) = v_0(x). \end{cases} \tag{1.1}$$

Here $v$ is the velocity field, $P$ is the pressure, both of which are the unknowns; $v_0$ is the given initial velocity; $\mu > 0$ is the viscosity constant, which will be taken as 1 unless

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stated otherwise. One can also add a forcing term on the righthand side, then it becomes a nonhomogeneous problem.

Thanks to Leray's work [62] in 1934, we know the above problem has a weak solution $v \in L^\infty((0,\infty), L^2(\mathbb{R}^3))$ such that $|\nabla v| \in L^2((0,\infty), L^2(\mathbb{R}^3))$ provided that the initial condition has finite kinetic energy. Moreover, $\|v(\cdot,t) - v_0(\cdot)\|_{L^2(\mathbb{R}^3)} \to 0$ as $t \to 0$ and $\forall T > 0$,

$$\int |v(x,T)|^2 dx + 2\int_0^T \int |\nabla v(x,t)|^2 dx dt \le \int |v_0(x)|^2 dx < \infty. \tag{1.2}$$

See Theorem 3.10 in Tsai's book [110] for a modern and concise proof for example. Solutions satisfying (1.2) are often referred to as Leray-Hopf solutions, in order to distinguish them from even weaker solutions. In general, one does not know if a Leray-Hopf solution is smooth, except for a few special cases, usually as a perturbation of a special smooth solution. Stability of NS under small perturbation is well studied. A general result of such kind can be found in [94] for example. Over the years several sufficient conditions under which Leray-Hopf solutions are smooth have been obtained. For example the Ladyzhenskaya-Prodi-Serrin condition: $|v| \in L^{p,q}_{x,t}$ with $\frac{3}{p} + \frac{2}{q} \le 1$ and $3 < p < \infty$ and the end point result $p = 3, q = \infty$ by Escauriaza, Seregin and Sverak [33]. See also [31] by Dong and Wang in higher dimensional cases, including both interior and boundary regularity. Here and later, a measurable function $f = f(x,t)$ is said to be in $L^{p,q}_{x,t}$ if

$$\|f\|_{L^{p,q}_{x,t}} \equiv \left( \int_0^\infty \left( \int_{\mathbb{R}^3} |f|^p dx \right)^{q/p} dt \right)^{1/q} < \infty.$$

If $\frac{3}{p} + \frac{2}{q} = 1$, these conditions are scaling invariant or critical under the natural scaling of the Navier Stokes equations: for $\lambda > 0$, if $(v, P)$ solves (1.1), then $(v_\lambda, P_\lambda)$ defined by

$$v_\lambda(x,t) \equiv \lambda v(\lambda x, \lambda^2 t), \quad P_\lambda(x,t) \equiv \lambda^2 P(\lambda x, \lambda^2 t), \tag{1.3}$$

also solves (1.1). It is easy to see that $\|v\|_{L^{p,q}_{x,t}} = \|v_\lambda\|_{L^{p,q}_{x,t}}$ for the above $p, q$. Sometimes these conditions can be improved logarithmically, even for endpoint cases. See the articles X. H. Pan [88], T. Tao [106], Barker and Prange [9], for example. A partial regularity result for the so-called "suitable weak solutions" was found by Caffarelli, Kohn and Nirenberg [17], building on earlier work of Scheffer [95, 96]. These solutions are Leray-Hopf solutions with an extra integrability condition on the pressure term $P$. It is proven that the singular set of suitable weak solutions, if exists, has one dimensional parabolic Hausdorff measure 0. The proof utilizes a blow up argument to deduce an $\epsilon$ regularity result: smallness of certain scaling invariant integral quantities involving the velocity or vorticity implies boundedness of solutions. Then the size estimate of the possible singular set follows from a covering argument. See also the papers of F. H. Lin [65], A. Vasseur [113], J. Wolf [117] for similar results and shorter proofs, some of which employ a De Giogi type (refined energy) method instead of blow up method. In [117], using the decomposition

$$(v \cdot \nabla) v = \frac{1}{2} \nabla |v|^2 - (\nabla \times v) \times v,$$