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Some Estimates for θ-type Calderón–Zygmund Operators and Linear Commutators on Certain Weighted Amalgam Spaces

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Abstract. In this paper, we first introduce some new kinds of weighted amalgam spaces. Then we discuss the strong type and weak type estimates for a class of Calderón–Zygmund type operators T_θ in these new weighted spaces. Furthermore, the strong type estimate and endpoint estimate of linear commutators $[b, T_\theta]$ formed by b and T_θ are established. Also we study related problems about two-weight, weak type inequalities for T_θ and $[b, T_\theta]$ in the weighted amalgam spaces and give some results.

Key Words: θ-type Calderón–Zygmund operators, commutators, weighted amalgam spaces, Muckenhoupt weights, Orlicz spaces.

AMS Subject Classifications: 42B20, 42B35, 46E30, 47B47

1 Introduction

Calderón–Zygmund singular integral operators and their generalizations on the Euclidean space $\mathbb{R}^n$ have been extensively studied (see [4, 11, 21, 24] for instance). In particular, Yabuta [24] introduced certain θ-type Calderón–Zygmund operators to facilitate his study of certain classes of pseudo-differential operators. Following the terminology of Yabuta [24], we introduce the so-called θ-type Calderón–Zygmund operators.

Definition 1.1. *Let θ be a non-negative, non-decreasing function on $\mathbb{R}^+ = (0, +\infty)$ with*

$$\int_0^1 \frac{\theta(t)}{t} dt < \infty. \tag{1.1}$$

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A measurable function $K(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$ is said to be a θ-type kernel if it satisfies

$$(i) \quad \big|K(x,y)\big| \le \frac{C}{|x-y|^n} \quad \text{for any} \quad x \neq y, \tag{1.2a}$$

$$(ii) \quad \big|K(x,y)-K(z,y)\big| + \big|K(y,x)-K(y,z)\big| \le \frac{C}{|x-y|^n} \cdot \theta\Big(\frac{|x-z|}{|x-y|}\Big) \quad \text{for } |x-z| < |x-y|/2. \tag{1.2b}$$

Definition 1.2. *Let T_θ be a linear operator from $\mathscr{S}(\mathbb{R}^n)$ into its dual $\mathscr{S}'(\mathbb{R}^n)$. We say that T_θ is a θ-type Calderón–Zygmund operator if*

1. *T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;*
2. *There is a θ-type kernel $K(x,y)$ such that*

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x,y) f(y) dy \tag{1.3}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and for all $x \notin \operatorname{supp} f$, where $C_0^\infty(\mathbb{R}^n)$ is the space consisting of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

Note that the classical Calderón–Zygmund operator with standard kernel (see [4,11]) is a special case of θ-type operator T_θ when $\theta(t) = t^\delta$ with $0 < \delta \le 1$.

Definition 1.3. *Given a locally integrable function b defined on $\mathbb{R}^n$, and given a θ-type Calderón–Zygmund operator T_θ, the linear commutator $[b, T_\theta]$ generated by b and T_θ is defined for smooth, compactly supported functions f as*

$$\begin{aligned}[b,T_\theta]f(x) :=& b(x)\cdot T_\theta f(x) - T_\theta(bf)(x) \\ =& \int_{\mathbb{R}^n} \big[b(x)-b(y)\big] K(x,y) f(y)\, dy. \end{aligned} \tag{1.4}$$

We first give the following weighted results of T_θ obtained by Quek and Yang in [19].

Theorem 1.1 ([19]). *Suppose that θ is a non-negative, non-decreasing function on $\mathbb{R}^+ = (0,+\infty)$ satisfying condition* (1.1). *Let $1 \le p < \infty$ and $w \in A_p$. Then the θ-type Calderón–Zygmund operator T_θ is bounded on $L^p_w(\mathbb{R}^n)$ for $p > 1$, and bounded from $L^1_w(\mathbb{R}^n)$ into $WL^1_w(\mathbb{R}^n)$ for $p = 1$.*

Since linear commutator has a greater degree of singularity than the corresponding θ-type Calderón–Zygmund operator, we need a slightly stronger condition (1.5) given below. The following weighted endpoint estimate for commutator $[b, T_\theta]$ of the θ-type Calderón–Zygmund operator was established in [26] under a stronger version of condition (1.5) assumed on θ, if $b \in BMO(\mathbb{R}^n)$ (for the unweighted case, see [15]). Let us