Analysis in Theory and Applications
Anal. Theory Appl., Vol. **38**, No. 4 (2022), pp. 394-416

DOI: 10.4208/ata.OA-2019-0037
December 2022

Some Generalized Clifford-Jacobi Polynomials and Associated Spheroidal Wavelets

Sabrine Arfaoui[1,3] and Anouar Ben Mabrouk[1,2,3,*]

[1] *Algebra, Number Theory and Nonlinear Analysis Laboratory LR18ES15, Department of Mathematics, Faculty of Sciences, Monastir 5000, Tunisia*
[2] *Department of Mathematics, Higher Institute of Applied Mathematics and Computer Science, Street of Assad Ibn Alfourat, Kairouan University, 3100 Kairouan Tunisia*
[3] *Department of Mathematics, Faculty of Sciences, University of Tabuk, Saudi Arabia*

Received 13 December 2019; Accepted (in revised version) 8 September 2020

Abstract. In the present paper, by extending some fractional calculus to the framework of Clifford analysis, new classes of wavelet functions are presented. Firstly, some classes of monogenic polynomials are provided based on 2-parameters weight functions which extend the classical Jacobi ones in the context of Clifford analysis. The discovered polynomial sets are next applied to introduce new wavelet functions. Reconstruction formula as well as Fourier-Plancherel rules have been proved. The main tool reposes on the extension of fractional derivatives, fractional integrals and fractional Fourier transforms to Clifford analysis.

Key Words: Continuous wavelet transform, Clifford analysis, Clifford Fourier transform, Fourier-plancherel, fractional Fourier transform, fractional derivatives, fractional integrals, fractional Clifford Fourier transform, Monogenic functions.

AMS Subject Classifications: 26A33, 42A38, 42B10, 44A15, 30G35

1 Introduction

Clifford Algebra is characterized by additional concepts as it provides a simpler model of mathematical objects compared to vector algebra. It permits a simplification in the notations of mathematical expressions such as plane and volume segments in two, three and higher dimensions by using a coordinate-free representation. Such representation is characterized by an important feature resumed in the fact that the motion of an object may be described with respect to a coordinate frame defined on the object itself. This means that it permits to use a self-coordinate system related to the object in hand.

*Corresponding author. *Email addresses:* `sabrine.arfaoui@issatm.rnu.tn` (S. Arfaoui), `anouar.benmabrouk@fsm.rnu.tn` (A. Ben Mabrouk)

In the present work, one aim is to provide a rigorous development of wavelets adapted to the sphere based on Clifford calculus. The frame is somehow natural as wavelets are characterized by scale invariance of approximation spaces. Clifford algebra is one mathematical object that owns this characteristic. Recall that multiplication of real numbers scales their magnitudes according to their position in or out from the origin. However, multiplication of the imaginary part of a complex number performs a rotation, it is a multiplication that goes round and round instead of in and out. So, a multiplication of spherical elements by each other results in an element of the sphere. Again, repeated multiplication of the imaginary part results in orthogonal components. Thus, we need a coordinates system that results always in the object, a concept that we will see again and again in the Algebra. In other words, Clifford algebra generalizes to higher dimensions by the same exact principles applied at lower dimensions, by providing an algebraic entity for scalars, vectors, bivectors, trivectors, and there is no limit to the number of dimensions it can be extended to. More details on Clifford analysis, Clifford calculus, origins, history, developments may be found in [1,13,15,26,29].

In the present work, we propose to develop new wavelet analysis constructed in the framework of Clifford analysis by adopting monogenic functions which may be described as solutions of the Dirac operator and are direct higher dimensional generalizations of holomorphic functions in the complex plane. We apply such extension to some well adapted Clifford weights to construct new spheroidal wavelets. Recall that wavelets are widespread in the last decades. They become an interesting and useful tool in many fields such as mathematics, quantum physics, electrical engineering and seismic geology and they have proved to meet a need in signal processing that Fourier transform was not the best answer. Classical Fourier analysis provides a global description of signals and did not provide a time localization.

Wavelet analysis starts by convoluting the analyzed function with copies $\psi_{a,b}$, $a > 0$ and $b \in \mathbb{R}$ (called wavelets) issued from a source (mother wavelet) ψ by dilation $a > 0$ and translation $b \in \mathbb{R}$,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\Big(\frac{x-b}{a}\Big). \tag{1.1}$$

Generally, the source ψ has to satisfy the so-called admissibility condition

$$\mathcal{A}_\psi = \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(u)|^2}{|u|} du < +\infty, \tag{1.2}$$

where $\widehat{\psi}$ is the Fourier transform of ψ.

Wavelet analysis of functions starts by computing the Continuous Wavelet Transform (CWT) of the analyzed function f

$$C_{a,b}(f) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\overline{\psi\Big(\frac{x-b}{a}\Big)} dx, \tag{1.3}$$