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Harmonic Analysis Associated with the Heckman-Opdam-Jacobi Operator on $\mathbb{R}^{d+1}$

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Abstract. In this paper we consider the Heckman-Opdam-Jacobi operator Δ_{HJ} on $\mathbb{R}^{d+1}$. We define the Heckman-Opdam-Jacobi intertwining operator V_{HJ}, which turn out to be the transmutation operator between Δ_{HJ} and the Laplacian Δ_{d+1}. Next we construct ${}^tV_{HJ}$ the dual of this intertwining operator. We exploit these operators to develop a new harmonic analysis corresponding to Δ_{HJ}.

Key Words: Heckman-Opdam-Jacobi operator, generalized intertwining operator and its dual, generalized Fourier transform, generalized translation operators.

AMS Subject Classifications: 33E30, 42B10, 44A15, 35K05

1 Introduction

Recently, Mejjaoli and Trimèche in [15], have considered and studied the Dunkl-Bessel-Laplace operator on $\mathbb{R}^d \times \mathbb{R}_+$ defined by

$$\Delta_{k,\beta} = \Delta_{k,x'} + L_{\beta,x_{d+1}}, \qquad x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}_+,$$

where Δ_k is the Dunkl Laplacian on $\mathbb{R}^d$ and L_β is the Bessel operator on $\mathbb{R}_+$ given by

$$L_\beta = \frac{d^2}{dx_{d+1}^2} + \frac{2\beta+1}{x_{d+1}} \frac{d}{dx_{d+1}}, \qquad \beta > -\frac{1}{2},$$

and have developed an harmonic analysis associated with it. Based on this paper, several works have been elaborated. We can cite, for example, the work of Hassini and Trimèche in [11]. They have studied the generalized wavelets and generalized windowed transforms associated to the Dunkl-Bessel operator.

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The Jacobi and the Cherednik Heckman-Opdam operators are differential and differential-difference operators defined respectively on $\mathbb{R}$ and $\mathbb{R}^d$, they are well studied in [1, 2, 4, 5, 12, 14, 17] and references there. These operators play a prominent role in the new harmonic analysis theory associated to a new class of differential-difference operators that we attempt to consider in this paper. This class of operators is given by

$$\Delta_{HJ} = \Delta_{k,x'} + (\Delta_{a,b} + \xi^2)_{x_{d+1}}, \qquad x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R},$$

where Δ_k is the Heckman-Opdam Laplacian on $\mathbb{R}^d$ (see [3, 9, 16, 18, 20]) and $\Delta_{a,b}$ is the Jacobi operator on $\mathbb{R}$ (see [1, 4, 5]).

Throughout this article, we have overcome several difficulties in proving some tools of harmonic analysis associated with the differential-difference operator Δ_{HJ} on $\mathbb{R}^{d+1}$ such that the Plancherel formula which is not verified but at the end we came up with an analog of it. The importance of this new class of operators is derived from those of the two operators Δ_k and $\Delta_{a,b}$.

The outline of this paper is as follows. The second and third sections are devoted to some basic results of harmonic analysis associated respectively with the Jacobi operator on $\mathbb{R}$ and the Cherednik operator on $\mathbb{R}^d$. In the last section, we study the harmonic analysis associated to Δ_{HJ}. In a more specific way, we give some properties of the eigenfunction Λ of this operator equal to 1 at zero. We introduce the generalized intertwining operator and its dual, we define the generalized Fourier transform and we prove for this transform the Paley-Wiener theorem and the inversion formulas. We finish by the generalized translation operators and convolution product that give us the Plancherel type formula. In a latest paper and as an application, we have solved the generalized heat equation associated to Δ_{HJ} and we have shown that the heat semi group has a positive kernel.

2 Harmonic analysis associated to the Jacobi operator on $\mathbb{R}$

In this section we recall some basic results that constitute the harmonic analysis for the Jacobi operator on $\mathbb{R}$. For more details we refer to [4–8, 18].

2.1 The Jacobi operator and function

For $a \geq b \geq -\frac{1}{2}$, $a \neq -\frac{1}{2}$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the Jacobi function $\varphi_\lambda^{(a,b)}$ is defined by

$$\varphi_\lambda(x) = \varphi_\lambda^{(a,b)}(x) = {}_2F_1\left(\frac{\xi + i\lambda}{2}; \frac{\xi - i\lambda}{2}; a+1; -(\sinh x)^2\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\xi = a + b + 1. \tag{2.1}$$