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Regularity of Viscosity Solutions of the Biased Infinity Laplacian Equation

Fang Liu[1,*], Fei Meng[2] and Xiaoyan Chen[1]

[1] *Department of Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China*
[2] *School of Science, Nanjing University of Posts and Telecommunications, Nanjing, Jiangsu 210003, China*

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Abstract. In this paper, we are interested in the regularity estimates of the nonnegative viscosity super solution of the $\beta-$biased infinity Laplacian equation

$$\Delta_\infty^\beta u = 0,$$

where $\beta \in \mathbb{R}$ is a fixed constant and $\Delta_\infty^\beta u := \Delta_\infty^N u + \beta|Du|$, which arises from the random game named biased tug-of-war. By studying directly the $\beta-$biased infinity Laplacian equation, we construct the appropriate exponential cones as barrier functions to establish a key estimate. Based on this estimate, we obtain the Harnack inequality, Hopf boundary point lemma, Lipschitz estimate and the Liouville property etc.

Key Words: $\beta-$biased infinity Laplacian, viscosity solution, exponential cone, Harnack inequality, Lipschitz regularity.

AMS Subject Classifications: 35J62, 35J70, 35B53

1 Introduction

In this work, we devote to the regularity of the viscosity solution of the $\beta-$biased infinity Laplacian equation

$$\Delta_\infty^\beta u = 0, \tag{1.1}$$

where $\beta \in \mathbb{R}$ ($\beta \neq 0$) is a fixed constant and

$$\Delta_\infty^\beta u := \Delta_\infty^N u + \beta|Du|$$

*Corresponding author. *Email addresses:* sdqdlf78@126.com, liufang78@njust.edu.cn (F. Liu), mengfei@njupt.edu.cn (F. Meng), cxy_161977@njust.edu.cn (X. Chen)

with

$$\Delta_\infty^N u := \frac{1}{|Du|^2}(D^2uDu)\cdot Du.$$

If $\beta=0$, it coincides with the unbiased operator.

The so-called $\beta-$biased infinity Laplacian operator Δ_∞^β is first introduced in [25] when modelling the biased tug-of-war. Let us briefly recall the two-player, random-turn, β-biased ε-tug-of-war game. Let F be a real final payoff function defined on $\partial\Omega$. The starting position is $x_0\in\Omega$. At the k-th step the two players toss a suitably biased coin (player I wins with odds of $\exp(\beta\varepsilon)$ to 1), and the winner chooses x_k with $d(x_k,x_{k-1})<\varepsilon$. The game ends when $x_k\in\partial\Omega$, and player II pays the amount $F(x_k)$ to player I. The existence and uniqueness of the viscosity solutions were obtained by game theory under the Dirichlet boundary condition. Furthermore, they also proved that viscosity solutions of (1.1) satisfies the comparison property with the exponential cones. For the general equations including (1.1), existence and uniqueness were proved by PDE methods under strong smoothness conditions on the boundary of the domain in [4]. For the inhomogeneous equation

$$\Delta_\infty^\beta u = f, \tag{1.2}$$

existence and uniqueness of Dirichlet or Dirichlet-Neumann mixed boundary problem were obtained by finite difference approximation [2]. In [20], the wellposedness and Lipschitz regularity of the initial-Dirichlet boundary problem related to the evolutionary equation

$$u_t-\Delta_\infty^\beta u = f,$$

were proven. Recently the infinity Laplacian equations arising from game theory have received a lot of attentions because they are not only degenerate from PDE point of view but also have many applications including the image processing, see for example [1, 10–12, 14, 18, 21, 22]. Notice that Δ_∞^β is 1-homogeneous and bounded when the gradient vanishes. This property of scaling invariance is very important in our proofs and applications including the image processing [10–12].

In the special case $\beta=0$, the unbiased operator is the well known normalized infinity Laplacian Δ_∞^N related to the absolutely minimizing Lipschitz extensions which has been extensively studied in the past two decades. See [3, 5, 7, 9, 23, 24, 26] and the references therein. Normalized infinity Laplacian equation was well studied by game theory named tug-of-war in [26] and by partial differential equation methods in [23] respectively. In [9], Crandall, Evans and Gariepy developed the method of comparison with cones for infinity harmonic functions. In [13], Harnack inequality was proven for the smooth nonnegative infinity harmonic functions. The Harnack inequality of the nonnegative infinity superharmonic functions(a supersolution to $\Delta_\infty^N u=0$) is obtained by $p-$approximation method in [19]. In [6], another proof was given using the distance functions as test functions.

In this paper, we are interested in the $\beta-$biased case and we focus on the $\beta-$biased operator itself. The trick is to construct suitable exponential cones as barrier functions