Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 1 (2023), pp. 1-15

DOI: 10.4208/ata.OA-2019-0023
March 2023

Multiple Integral Inequalities for Schur Convex Functions on Symmetric and Convex Bodies

Silvestru Sever Dragomir[1,2,*]

[1] *Mathematics, College of Engineering & Science, Victoria University, Melbourne City, MC 8001, Australia*
[2] *DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa*

Received 18 July 2019; Accepted (in revised version) 31 January 2021

Abstract. In this paper, by making use of Divergence theorem for multiple integrals, we establish some integral inequalities for Schur convex functions defined on bodies $B \subset \mathbb{R}^n$ that are symmetric, convex and have nonempty interiors. Examples for three dimensional balls are also provided.

Key Words: Schur convex functions, multiple integral inequalities.

AMS Subject Classifications: 26D15

1 Introduction

For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq \cdots \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = \left(x_{[1]}, \cdots, x_{[n]}\right)$ denote the decreasing rearrangement of x. For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1, \cdots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be majorized by y (y majorizes x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps Schur-increasing would be more appropriate, but the term Schur-convex is by now well entrenched in the literature, [5, p. 80].

*Corresponding author. *Email address:* `sever.dragomir@vu.edu.au` (S. Dragomir)

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if

$$x \prec y \quad \text{on } \mathcal{A} \Rightarrow \phi(x) \le \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on A. If $A = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [5] and the references therein. For some recent results, see [2–4] and [6–8].

The following result is known in the literature as Schur-Ostrowski theorem [5, p. 84]:

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n \tag{1.2}$$

and for all $i \neq j$, with $i, j \in \{1, \cdots, n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \ge 0 \quad \text{for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

With the aid of (1.2), condition (1.3) can be replaced by the condition

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \ge 0 \quad \text{for all } z \in I^n. \tag{1.4}$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is symmetric in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π;

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [5, p. 85].

Theorem 1.2. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.5}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \ge 0 \quad \text{for all } z \in \mathcal{A}. \tag{1.6}$$