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Existence of Solution for a General Class of Strongly Nonlinear Elliptic Problems Having Natural Growth Terms and L^1-Data

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Abstract. This paper is concerned with the existence of solution for a general class of strongly nonlinear elliptic problems associated with the differential inclusion

$$\beta(u) + A(u) + g(x,u,Du) \ni f,$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, β maximal monotone mapping such that $0 \in \beta(0)$, while $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign-condition on s. The right hand side f is assumed to belong to $L^1(\Omega)$.

Key Words: Sobolev spaces, Leray-Lions operator, trunctions, maximal monotone graphe.

AMS Subject Classifications: 35J15, 35J20, 35J60

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence of solutions for the following strongly nonlinear elliptic inclusion

$$(E,f) \quad \begin{cases} \beta(u) + A(u) + g(x,u,Du) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), \quad g(x,u,Du) \in L^1(\Omega), \end{cases}$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ $(1 < p < \infty)$ defined as $A(u) = -div(a(x,u,Du))$, β maximal montone mapping such that $0 \in \beta(0)$, g is a nonlinear lower term having "natural growth" (of order p) with respect to Du, with

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respect to u, we do not assume any growth restrictions, but it satisfies a "sign-condition" on s and $f \in L^1(\Omega)$.

It will turn out that, for each solution u, $g(x,u,Du)$ will be in $L^1(\Omega)$, but for each $v \in W_0^{1,p}(\Omega)$, $g(x,v,Dv)$ can be very odd, and does not necesserily belong to $W^{-1,p'}(\Omega)$.

Particular instances of problem (E,f) have been studied for $\beta = 0$, Boccardo, Gallouët and Murat in [6] have proved the existence of at least one solution for the problem. Let us point out that another work in this direction can be found in [4].

Another important work in the L^1-theory for p-Laplacian type equations is [3] where problem

$$\begin{cases} -div(a(x,Du)) + \beta(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [1], Y.Akdim and C.Allalou have proved the existence of renormalized solution for an elliptic problem type diffusion-convection in the framework of weighted variable exponent Sobolev spaces

$$(E) \quad \begin{cases} \beta(u) - div(a(x,Du) + F(u)) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We also refer to [10,13], For results on the existence of renormalized solutions of elliptic problems of type (E).

The present paper is organized as follows: in Section 2, we give basic assumptions on a, g, β and f. In Section 3, we study our main result, existence of solution to (E,f) for any L^1-data f. To prove the main result, we will introduce and solve, in Section 4, approximating problems for any L^∞-data f. The proof of main result is given in Section 5. The last section is devoted to an example for illustrating our abstract result.

2 Assumptions

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence of solution to the strongly nonlinear elliptic inclusion problem with Dirichlet boundary conditions

$$(E,f) \quad \begin{cases} \beta(u) + A(u) + g(x,u,Du) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), & g(x,u,Du) \in L^1(\Omega), \end{cases}$$

with right-hand side $f \in L^1(\Omega)$. A is a non linear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) defined by

$$A(u) = -div(a(x,u,Du)),$$