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Busemann-Petty Type Problem for the General L_p-Centroid Bodies

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Abstract. Lutwak showed the Busemann-Petty type problem (also called the Shephard type problem) for the centroid bodies. Grinberg and Zhang gave an affirmation and a negative form of the Busemann-Petty type problem for the L_p-centroid bodies. In this paper, we obtain an affirmation form and two negative forms of the Busemann-Petty type problem for the general L_p-centroid bodies.

Key Words: L_p-centroid body, general L_p-centroid body, Busemann-Petty problem, affirmation form, negation form.

AMS Subject Classifications: 52A40, 52A20, 52A39, 52A38

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in n-dimensional Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let $\mathcal{S}^n_o$ and $\mathcal{S}^n_{os}$ orderly denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of a body K, for the standard unit ball B in $\mathbb{R}^n$, write $\omega_n = V(B)$.

Centroid body was attributed by Blaschke to Dupin (see [6, 18]), its definition was extended by Petty (see [17]). Let K is a compact set, the centroid body, ΓK, of K is an origin-symmetric convex body whose support function is given by (see [6])

$$h_{\Gamma K}(u) = \frac{1}{V(K)} \int_K |u \cdot x| dx \tag{1.1}$$

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for all $u \in S^{n-1}$.

Centroid bodies are very important in Brunn-Minkowski theory. For decades, centroid bodies have attracted increased attention (for example see articles [10,11,17,27] and books [6, 18]). In particular, Lutwak [11] showed an affirmation and a negative form of the Busemann-Petty type problems for the centroid bodies as follows:

Theorem 1.1. *For $K \in \mathcal{S}_o^n$, $L \in \mathcal{P}^*$, if $\Gamma K \subseteq \Gamma L$, then*

$$V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$. Here $\mathcal{P}^*$ denotes the set of polars of all projection bodies.

Theorem 1.2. *If $K \in \mathcal{S}_{os}^n \backslash \mathcal{P}^*$ is infinite smooth, then there exists $L \in \mathcal{S}_{os}^n \backslash \mathcal{P}^*$ is infinite smooth, such that $\Gamma K \subset \Gamma L$, but*

$$V(K) > V(L).$$

In 1997, Lutwak and Zhang [15] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) =& \frac{1}{c_{n,p} V(K)} \int_K \mid u \cdot x \mid^p dx \\ =& \frac{1}{c_{n,p}(n+p) V(K)} \int_{S^{n-1}} \mid u \cdot v \mid^p \rho_K(v)^{n+p} dv \end{aligned} \tag{1.2}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_n \omega_{p-1} \tag{1.3}$$

and dv is the standard spherical Lebesgue measure on S^{n-1}. The normalization above is chosen so that for the standard unit ball B in $\mathbb{R}^n$, we have $\Gamma_p B = B$. For the case $p = 1$, by (1.1) and (1.2), we see that $\Gamma_1 K$ is the centroid body ΓK under the normalization of definition (1.2) and $\Gamma_1 K = c_{n,1}^{-1} \Gamma K$ (see [6]).

Further, Lutwak and Zhang [15] established the L_p-centroid affine inequality. Whereafter, associated with the L_p-centroid bodies, Lutwak, Yang and Zhang [14] proved the L_p-Busemann-Petty centroid inequality which is stronger than the L_p-centroid affine inequality. The L_p-centroid bodies mean that the centroid bodies are extended from the Brunn-Minkowski theory to the L_p-Brunn-Minkowski theory. Regarding the studies of the L_p-centroid bodies, also see [1–3,7,21,22,24] and books [6,18]. In particular, Grinberg and Zhang [7] gave the following the Busemann-Petty type problem for the L_p-centroid bodies.

Theorem 1.3. *If $K \in \mathcal{S}_o^n$, $L \in \mathcal{P}_p^*$, then $\Gamma_p K \subseteq \Gamma_p L$ implies*

$$V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$. Here $\mathcal{P}_p^*$ denotes the set of polars of all L_p-projection bodies.