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Characterizations of Bounded Singular Integral Operators on the Fock Space and Their Berezin Transforms

Xingtang Dong[1,*] and Li Feng[1]

School of Mathematics, Tianjin University, Tianjin 300350, China

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Abstract. There is a singular integral operators S_φ on the Fock space $\mathcal{F}^2(\mathbb{C})$, which originated from the unitarily equivalent version of the Hilbert transform on $L^2(\mathbb{R})$. In this paper, we give an analytic characterization of functions φ with finite zeros such that the integral operator S_φ is bounded on $\mathcal{F}^2(\mathbb{C})$ using Hadamard's factorization theorem. As an application, we obtain a complete characterization for such symbol functions φ such that the Berezin transform of S_φ is bounded while the operator S_φ is not. Also, the corresponding problem in higher dimensions is considered.

Key Words: Fock space, singular integral operators, boundedness, Berezin transform.

AMS Subject Classifications: 30H20, 47G10

1 Introduction

The Fock space $\mathcal{F}^2(\mathbb{C}^n)$ is the Hilbert space of all entire functions f on the complex Euclidean space $\mathbb{C}^n$ such that

$$\|f\|^2_{\mathcal{F}^2(C^n)} = \int_{\mathbb{C}^n} |f(z)|^2 d\lambda(z) < \infty,$$

where

$$d\lambda(z) = \pi^{-n} e^{-|z|^2} dz$$

is the Gaussian measure on $\mathbb{C}^n$. The inner product on $\mathcal{F}^2(\mathbb{C}^n)$ is inherited from $L^2(\mathbb{C}^n, d\lambda)$. This space is convenient setting for many problems in functional analysis, mathematical physics and engineering. We refer the interested reader to [9] for the mathematical theory of the Fock space and [8] for applications of the Fock space in physics.

*Corresponding author. *Email addresses:* `dongxingtang@163.com` (X. Dong), `fengli@163.com` (L. Feng)

The Bargmann transform B is the operator from $L^2(\mathbb{R}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ defined by

$$Bf(z) = c\int_{\mathbb{R}^n} f(x)e^{2x\cdot z - x^2 - (z^2/2)}dx,$$

where $c = (2/\pi)^{n/4}$ and $x \cdot z = x_1z_1 + \cdots + x_nz_n$. The Bargmann transform is an old tool in mathematics analysis and mathematical physics. See [1,2,9,11] and references therein.

It is well known that B is a unitary operator from $L^2(\mathbb{R}^n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$. Hilbert transform is a typical bounded "singular integral operator" on $L^2(\mathbb{R})$ and is one of the most studied objects in harmonic analysis. Motivated by the unitarily equivalent version of such an operator on $\mathcal{F}^2(\mathbb{C})$, Zhu [10] proposed the following open problem for the Fock space.

(Q1). Characterize those symbol functions $\varphi \in \mathcal{F}^2(\mathbb{C})$ such that the integral operator

$$S_\varphi F(z) = \int_{\mathbb{C}} F(w)e^{z\overline{w}}\varphi(z-\overline{w})d\lambda(w) \tag{1.1}$$

is bounded on $\mathcal{F}^2(\mathbb{C})$.

This problem has attracted the attention of several analysts in recent years. In particular, Cao et al. [4] obtained a complete solution to this open problem for the Fock space in all dimensions using harmonic analysis methods. Indeed, they got the following result on the Fock space in $\mathbb{C}^n$.

Theorem 1.1 ([4, Theorem 1.1]). *For $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$, the integral operator S_φ in* (4.1) *is bounded on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if there exists an $m \in L^\infty(\mathbb{R}^n)$ such that*

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{i}{2}z)^2}dx, \qquad z \in \mathbb{C}^n. \tag{1.2}$$

Moreover,

$$\|S_\varphi\|_{\mathcal{F}^2(\mathbb{C}^n)\to\mathcal{F}^2(\mathbb{C}^n)} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$

For a given symbol function $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$, even though [4, Proposition 3.6] also shows how φ gives rise to m, it is difficult to check directly whether the integral operator S_φ is bounded or not. In this paper, we aim to give a direct analytic characterization of functions $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$ such that the integral operator S_φ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.

Our first result focus on the corresponding problem for the symbol functions with a finite number of zeros on the Fock space over the complex plane.

Theorem 1.2. *Let $\varphi \in \mathcal{F}^2(\mathbb{C})$ with only a finite number of zeros. Then the operator S_φ in* (1.1) *is bounded on $\mathcal{F}^2(\mathbb{C})$ if and only if one of the following conditions holds:*

(1) $\varphi(z) = ce^{az^2+bz}$ *for some constant $b \in \mathbb{R}$ and constants $a, c \in \mathbb{C}$ such that $c \neq 0$, $a \neq \frac{1}{2}$ and* $2|a|^2 = \Re a$.