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Interior Gradient Estimates for General Prescribed Curvature Equations

Zhenan Sui[1] and Wei Sun[2,*]

[1] *Institute for Advanced Study in Mathematics of HIT, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China*
[2] *Institute of Mathematical Sciences, ShanghaiTech University, Shanghai 201210, China*

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Abstract. In this paper, we derive the interior gradient estimate for solutions to general prescribed curvature equations. The proof is based on a fundamental observation of Gårding's cone and some delicate inequalities under a suitably chosen coordinate chart. As an application, we obtain a Liouville type theorem.

Key Words: Interior gradient estimate, prescribed curvature equations.

AMS Subject Classifications: 53C21

1 Introduction

In this paper, we consider the general prescribed curvature equations

$$F(A) = f(\boldsymbol{\lambda}(A)) = \psi(x), \tag{1.1}$$

where ψ is a prescribed C^1 function defined on some domain in $\mathbb{R}^n$, $\boldsymbol{\lambda}(A) = (\lambda_1, \cdots, \lambda_n)$ is the vector of eigenvalues of the Weingarten curvature matrix A with entries

$$a_{ij} = \frac{u_{ij}}{w} - \sum_k \frac{u_i u_k u_{kj}}{w^2(1+w)} - \sum_k \frac{u_j u_k u_{ik}}{w^2(1+w)} + \sum_{k,l} \frac{u_i u_j u_k u_l u_{kl}}{w^3(1+w)^2} \tag{1.2}$$

and $w = \sqrt{1+|Du|^2}$. These eigenvalues are known as the principal curvatures of the vertical graph of u in $\mathbb{R}^{n+1}$ with respect to its upward normal vector field ν.

Prescribed curvature equations were first studied by Caffarelli-Nirenberg-Spruck [3, 4], and interior gradient estimate is a crucial step in the study of existence and regularity of prescribed hypersurfaces. It can be used to establish existence results for Dirichlet

*Corresponding author. *Email addresses:* `sunwei@shanghaitech.edu.cn` (W. Sun), `sui.4@osu.edu` (Z. Sui)

problems with less regular boundary data or seek locally Lipschitz continuous viscosity solutions on noncompact domains. For the significance of C^1 estimates (versus C^2 estimates), one can see from the σ_k Loewner-Nirenberg problem that, for $2 \leq k \leq n$, there are nonexistence results of C^2 solutions (see [12,13]), so no C^2 solutions is available (if C^2 estimates hold, higher derivative estimates hold, and one would obtain smooth solution). The existence of Lipschitz continous viscosity solutions to the σ_k Loewner-Nirenberg problem relies on C^1 estimates (see [6]). It is reasonable to expect that for a large class of equations of this type (a class of augmented Hessian equations) there are such phenomena, which need further study. Due to the importance of C^1 estimates, especially for the interior gradient estimate for general prescribed curvature equations, our aim is to derive such an estimate using PDE method.

In [10], Li initiated the study of interior gradient estimate for general prescribed curvature equations, where f is a smooth symmetric function, defined on an open symmetric convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive cone

$$\Gamma_n = \big\{\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n | \lambda_i > 0,\ i = 1, \ldots, n\big\}.$$

In addition, f is assumed to satisfy

$$f_i := \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma \quad \text{for } i = 1, \ldots, n, \tag{1.3a}$$

$$f \ \text{ is concave in } \Gamma, \tag{1.3b}$$

letting $\psi \geq \psi_0$ for some positive constant ψ_0, and assume also that there exist $c_0, c_1 > 0$, which depends on ψ_0, such that, for any $\boldsymbol{\lambda} \in \Gamma$ with $f(\boldsymbol{\lambda}) \geq \psi_0$, $\lambda_i \leq 0$,

$$f_i \geq c_0 \sum_{j \neq i} f_j + c_1, \tag{1.4a}$$

$$\limsup_{\substack{\boldsymbol{\lambda} \in \Gamma \\ \boldsymbol{\lambda} \to \boldsymbol{\lambda_0}}} f(\boldsymbol{\lambda}) < \psi_0, \quad \forall \boldsymbol{\lambda_0} \in \partial\Gamma, \tag{1.4b}$$

$$\liminf_{\substack{\boldsymbol{\lambda} \in \Gamma \\ \boldsymbol{\lambda} \to \mathbf{0}}} f(\boldsymbol{\lambda}) > -\infty. \tag{1.4c}$$

Typical examples of f satisfying the above conditions are $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ defined on Γ_k, $0 \leq l < k \leq n$, where $\sigma_0 = 1$ and for $k = 1, \ldots, n$,

$$\sigma_k(\boldsymbol{\lambda}) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

which is the kth elementary symmetric function defined on kth Gårding's cone

$$\Gamma_k = \big\{\boldsymbol{\lambda} \in \mathbb{R}^n | \sigma_i(\boldsymbol{\lambda}) > 0,\ i = 1, \ldots, k\big\},$$

(see [2–4] for the verification of (1.3a), (1.3b) and [14] for (1.4a)).