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A Remark about Time-Analyticity of the Linear Landau Equation with Soft Potential

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Abstract. In this note, we study the Cauchy problem of the linear spatially homogeneous Landau equation with soft potentials. We prove that the solution to the Cauchy problem enjoys the analytic regularizing effect of the time variable with an L^2 initial datum for positive time. So that the smoothing effect of Cauchy problem for the linear spatially homogeneous Landau equation with soft potentials is similar to the heat equation.

Key Words: Spatially homogeneous Landau equation, analytic smoothing effect, soft potentials.

AMS Subject Classifications: 35B65, 76P05, 82C40

1 Introduction

The Cauchy problem of spatially homogenous Landau equation reads

$$\begin{cases} \partial_t F = Q(F,F), \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F = F(t,v) \geq 0$ is the density distribution function at time $t \geq 0$, with the velocity variable $v \in \mathbb{R}^3$. The Landau bilinear collision operator is defined by

$$Q(G,F)(v) = \sum_{j,k=1}^{3} \partial_j \bigg(\int_{\mathbb{R}^3} a_{jk}(v-v_*)[G(v_*)\partial_k F(v) - \partial_k G(v_*)F(v)]dv_* \bigg) \tag{1.2}$$

with

$$a_{jk}(v) = (\delta_{jk}|v|^2 - v_j v_k)|v|^\gamma, \quad \gamma \geq -3,$$

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is a symmetric non-negative matrix such that

$$\sum_{j,k=1}^{3} a_{jk}(v)v_j v_k = 0.$$

Here, γ is a parameter which leads to the classification of the hard potential if $\gamma > 0$, Maxwellian molecules if $\gamma = 0$, soft potential if $-3 < \gamma < 0$ and Coulombian potential if $\gamma = -3$.

The Landau equation was introduced as a limit of the Boltzmann equation when the collisions become grazing in [6, 18]. In the hard potential case, the existence, and the uniqueness of the solution to the Cauchy problem for the spatially homogeneous Landau equation have been addressed by Desvillettes and Villani in [7,19]. Meanwhile, they also proved the smoothness of the solution is $C^\infty(]0,\infty[;\mathcal{S}(\mathbb{R}^3))$. The analytic and the Gevrey regularity of the solution for any $t > 0$ have already been studied in [1,2].

We shall study the linearization of the Landau equation (1.1) near the Maxwellian distribution

$$\mu(v) = (2\pi)^{\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Considering the fluctuation of the density distribution function

$$F(t,v) = \mu(v) + \sqrt{\mu}(v) f(t,v),$$

since $Q(\mu,\mu) = 0$, the Cauchy problem (1.1) takes the form

$$\begin{cases} \partial_t f + \mathcal{L} f = \Gamma(f,f), \\ f|_{t=0} = f_0, \end{cases}$$

with $F_0 = \mu + \sqrt{\mu} f_0$, where

$$\begin{aligned} &\Gamma(f,f) = \mu^{-\frac{1}{2}} Q(\sqrt{\mu} f, \sqrt{\mu} f), \\ &\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad \text{with } \mathcal{L}_1 f = -\Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 f = -\Gamma(f, \sqrt{\mu}). \end{aligned}$$

The spatially homogeneous Landau equation and non-cutoff Boltzmann equation in a close-to-equilibrium framework have been studied in [10] and the Gelfand-Shilov smoothing effect has been proved in [11, 15]. Guo [8] constructed the classical solution for the spatially inhomogeneous Landau equation near a global Maxwellian in a periodic box. The smoothness of the solutions has been studied in [3, 9, 12]. In addition, the analytic smoothing effect of the velocity variable for the nonlinear Landau equation has been treated in [13,16]. The variant regularity results under a close-to-equilibrium setting have been considered in [4,5,17].

In this work, we consider the Cauchy problem of the linear Landau equation, such as

$$\begin{cases} \partial_t f + \mathcal{L} f = g, \\ f|_{t=0} = f_0, \end{cases} \tag{1.3}$$