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Bilinear Pseudo-Differential Operator and Its Commutator on Generalized Fractional Weighted Morrey Spaces

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Abstract. The aim of this paper is to establish the boundedness of bilinear pseudo-differential operator T_σ and its commutator $[b_1, b_2, T_\sigma]$ generated by T_σ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ on generalized fractional weighted Morrey spaces $L^{p,\eta,\varphi}(\omega)$. Under assumption that a weight satisfies a certain condition, the authors prove that T_σ is bounded from products of spaces $L^{p_1,\eta_1,\varphi}(\omega_1) \times L^{p_2,\eta_2,\varphi}(\omega_2)$ into spaces $L^{p,\eta,\varphi}(\vec{\omega})$, where $\vec{\omega} = (\omega_1, \omega_2) \in A_{\vec{P}}$, $\vec{P} = (p_1, p_2)$, $\eta = \eta_1 + \eta_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p_1, p_2 \in (1, \infty)$. Furthermore, the authors show that the $[b_1, b_2, T_\sigma]$ is bounded from products of generalized fractional Morrey spaces $L^{p_1,\eta_1,\varphi}(\mathbb{R}^n) \times L^{p_2,\eta_2,\varphi}(\mathbb{R}^n)$ into $L^{p,\eta,\varphi}(\mathbb{R}^n)$. As corollaries, the boundedness of the T_σ and $[b_1, b_2, T_\sigma]$ on generalized weighted Morrey spaces $L^{p,\varphi}(\omega)$ and on generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ is also obtained.

Key Words: Generalized fractional weighted Morrey space, bilinear pseudo-differential operator, commutator, space $\mathrm{BMO}(\mathbb{R}^n)$.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

In 1967, Hörmander first introduced the definition of a pseudo-differential operator (see [13]), that is, let $\sigma(x, \xi)$ be a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$, then the pseudo-differential operator $\widetilde{T}_\sigma$ is defined by

$$\widetilde{T}_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) \widehat{f}(\xi) e^{ix\cdot\xi} d\xi \quad \text{for } f \in \mathcal{S}, \tag{1.1}$$

where $\widehat{f}$ represents the Fourier transform of f, and the smooth function σ belongs to the symbol classes $S^m_{\rho,\delta}$, which consist of all σ with satisfying the differential inequality

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

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for multi-indices $\alpha,\beta \in \mathbb{N}^n$, where $m \in \mathbb{R}$ and $0 \leq \rho,\delta \leq 1$. Such operators not only generalize the definition of differential operators with variable coefficients, but also have a key application in PDE. Therefore, the study of the pseudo-differential operator $\widetilde{T}_\sigma$ is widely focused. For example, Calderón and Vaillancourt in [5] proved that $\widetilde{T}_\sigma$ is bounded on space $L^2(\mathbb{R}^n)$. In 1988, Cardery and Seeger obtained the boundedness of pseudo-differential operator $\widetilde{T}_\sigma$ on spaces L^p (see [4]). The more researches about the pseudo-differential operators $\widetilde{T}_\sigma$ on various of function spaces can be seen [1,2,10,11,14] and the references therein.

However, in 1975, Coifman and Meyer obtained the definition of bilinear pseudo-differential operators and their some properties (see [8]). Namely, let $m \in \mathbb{R}$ and $\rho,\delta \in [0,1]$. A symbol in $BS^m_{\rho,\delta}$ is a smooth function $\sigma(x,\xi,\eta)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices $\alpha,\beta,\gamma \in \mathbb{N}^n$, the following inequality

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m-\rho(|\beta|+|\gamma|)+\delta|\alpha|}$$

holds. Respectively, the bilinear pseudo-differential operators T_σ associated with the above function $\sigma(x,\xi,\eta) \in BS^m_{\rho,\delta}$ is defined by

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \quad \text{for } f_1,f_2 \in \mathcal{S}. \tag{1.2}$$

In this paper, we will mainly consider the symbol $\sigma(x,\xi,\eta) \in BS^0_{1,0}$, that is,

$$\begin{aligned}&|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \\ \leq& C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{-(|\beta|+|\gamma|)} \quad \text{for all multi-indices } \alpha,\beta,\gamma \in \mathbb{N}^n.\end{aligned} \tag{1.3}$$

If we denote $\kappa(x,y,z)$ by the inverse Fourier transform (in the ξ-variable and η-variable) of the function $\sigma(x,\xi,\eta)$ (i.e., $\kappa(x,y,z) = \mathcal{F}_\xi^{-1}\mathcal{F}_\eta^{-1}\sigma(x,\xi,\eta)$), then

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \kappa(x,y,z)f_1(x-y)f_2(x-z)dydz. \tag{1.4}$$

Further, if we set $K(x,y,z) = \kappa(x,x-y,x-z)$, then the bilinear pseudo-differential operators T_σ defined as in (1.4) is changed into the following standard form

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z)f_1(y)f_2(z)dydz. \tag{1.5}$$

Since then, the research about T_σ defined as in (1.5) on various function spaces is widely focused. For example, Bényi and Torres proved that T_σ is bounded from the products of spaces $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for all $1 < p,q < \infty$ (see [3]). In 2012, Xiao et al. [26] showed that T_σ is bounded on the products of local Hardy spaces. More researches on the bilinear pseudo-differential operators can be seen [18–20, 25].

Before stating the organization of this paper, we first recall the definition of bound mean oscillation space $=$ BMO$(\mathbb{R}^n)$) in [15].