Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 2 (2024), pp. 111-171

DOI: 10.4208/ata.OA-2023-0015
June 2024

Hölder Regularity for the Linearized Porous Medium Equation in Bounded Domains

Tianling Jin[1,*] and Jingang Xiong[2]

[1] *Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*
[2] *School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE, Beijing Normal University, Beijing 100875, China*

Received 7 July 2023; Accepted (in revised version) 13 January 2024

Abstract. In this paper, we systematically study weak solutions of a linear singular or degenerate parabolic equation in a mixed divergence form and nondivergence form, which arises from the linearized fast diffusion equation and the linearized porous medium equation with the homogeneous Dirichlet boundary condition. We prove the Hölder regularity of their weak solutions.

Key Words: Singular parabolic equations, degenerate parabolic equations, regularity, porous medium equations.

AMS Subject Classifications: 35B65, 35K20, 35J75

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a smooth bounded open set, and ω be a smooth function in $\overline{\Omega}$ comparable to the distance function $d(x) := \mathrm{dist}(x, \partial\Omega)$, that is,

$$0 < \inf_{\Omega} \frac{\omega}{d} \leq \sup_{\Omega} \frac{\omega}{d} < \infty.$$

For example, ω can be taken as the positive normalized first eigenfunction of $-\Delta$ in Ω with Dirichlet zero boundary condition. Let

$$p > -1 \tag{1.1}$$

be a fixed constant throughout the paper unless otherwise stated.

*Corresponding author. *Email addresses:* tianlingjin@ust.hk (T. Jin), jx@bnu.edu.cn (J. Xiong)

In this paper, we would like to study regularity of weak solutions to

$$\begin{aligned} &a\omega^p\partial_t u - D_j(a_{ij}D_i u + d_j u) + b_i D_i u + \omega^p c u + c_0 u \\ &= \omega^p f + f_0 - D_i f_i \qquad \text{in } \Omega\times(-1,0], && (1.2a)\\ &u=0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega\times(-1,0], && (1.2b)\end{aligned}$$

where all a, a_{ij}, d_j, b_i, c, c_0, f, f_0, f_i are functions of (x,t), $D_i=\partial_{x_i}$, and the summation convention is used. Throughout this paper, we always assume the ellipticity condition, that is, (a_{ij}) is a matrix satisfying

$$\forall (x,t)\in\Omega\times[-1,0], \qquad \lambda\le a(x,t)\le\Lambda, \tag{1.3a}$$

$$\lambda|\xi|^2\le\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j\le\Lambda|\xi|^2, \qquad \forall\xi\in\mathbb{R}^n, \tag{1.3b}$$

where $0<\lambda\le\Lambda<\infty$.

The study of Eqs. (1.2) is motivated by the linearized equation of the fast diffusion equations (corresponding to $p>0$ in (1.4)) or slow diffusion equations (corresponding to $-1<p<0$ in (1.4), which are also called porous medium equations)

$$\partial_t v^{p+1}=\Delta v \qquad \text{in } \Omega\times(0,\infty), \tag{1.4a}$$

$$v=0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{1.4b}$$

From DiBenedetto-Kwong-Vespri [7], we know that the solution v of (1.4) with $p>0$ satisfies the global Harnack inequality

$$0<\inf_{\Omega}\frac{v(t,x)}{d(x)}\le\sup_{\Omega}\frac{v(t,x)}{d(x)}<\infty \tag{1.5}$$

before its extinction time. See Bonforte-Figalli [2] for a survey. From Aronson-Peletier [1], we also know that the solution v of (1.4) with $-1<p<0$ satisfies (1.5) as well after certain waiting time. Therefore, the linearized equation of (1.4), which plays an important role in proving optimal regularity of solutions to (1.4) in [19–21], falls into a form of the Eq. (1.2). In our earlier work [19], we have obtained many properties for equations like (1.2) with $p>0$, such as well-posedness, local boundedness and Schauder estimates. In this paper, we study the Eq. (1.2) in a more general and systematic way. The main goal of this paper is the Hölder regularity of its weak solutions to (1.2) up to the boundary $\{x_n=0\}$.

After the De Giorgi-Nash-Moser theory on the Hölder regularity for uniformly elliptic and uniformly parabolic equations, there have been many investigations on regularity for degenerate or singular elliptic and parabolic equations. By the work of Fabes-Kenig-Serapioni [13], we still have Hölder regularity for elliptic equations whose coefficients are of A_2 weight. See also earlier work of Kruzkov [23], Murthy-Stampacchia [24],