Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 2 (2024), pp. 172-190

DOI: 10.4208/ata.OA-2022-0022
June 2024

The Weighted L^p-Boundedness for a Class of Multilinear Oscillatory Singular Integrals Related to Block Spaces

Huoxiong Wu* and Weijin Yan

School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China

Received 16 July 2022; Accepted (in revised version) 23 March 2024

Abstract. This paper obtains a criterion on the weighted L^p-boundedness for a class of multilinear oscillatory singular operators with real-valued polynomial phases and rough kernels belonging to the block spaces.

Key Words: Multilinear oscillatory integrals, block space, BMO function, radial weights.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

We will work on $\mathbb{R}^n, n \geq 2$. Let Ω be a homogeneous function of degree zero with mean value zero on the unit sphere S^{n-1}. Define the oscillatory singular integral operator T by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. In 1987, Ricci and Stein [14] first studied the L^p boundedness for T with smooth kernel. In 1992, Lu and Zhang [13] extended the result of [14] to rough kernel case and established a simple criterion for L^p-boundedness of these operators. In 1998, Chen, Hu and Lu [1] studied the multilinear oscillatory integral operator defined by

$$T_A f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)dy,$$

where $\Omega \in L^q(S^{n-1})$ for some $q > 1$, $P(x,y)$ is a real-valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $R_{m+1}(A;x,y)$ denotes the $(m+1)$-th $(m \geq 1)$ remainder of the Taylor series

*Corresponding author. *Email addresses:* huoxwu@xmu.edu.cn (H. Wu), weijinyan0408@163.com (W. Yan)

of A at x about y, more precisely,

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\gamma|\le m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma,$$

and $D^\gamma A \in \mathrm{BMO}(\mathbb{R}^n)$ for all multi-indices $|\gamma| = m$. They showed that for any nontrivial real-valued polynomial $P(x,y)$, the L^p-boundedness of T_A is equivalent to the corresponding L^p-boundedness of S_A, the truncated operator of T_A without phase. Subsequently, Ding, Lu and Yang [6] extended the result to the weighted case.

On the other hand, to study the boundedness of singular integrals with rough kernels, Jiang and Lu [9] introduced in the block space $B_q^{0,v}(S^{n-1})$ $(q>1,\ v>-1)$, and Keitoku and Sato [10] pointed out that

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,v_1}(S^{n-1}) \subset B_q^{0,v_2}(S^{n-1}), \quad \forall -1 < v_2 < v_1,$$

which are proper inclusions. Also from [10], we know that $B_q^{0,0}(S^{n-1})$ is not contained in $L(\log^+ L)^{1+\varepsilon}(S^{n-1})$ for any $\varepsilon > 0$ although the relationship between $B_q^{0,0}(S^{n-1})$ and $L\log^+ L(S^{n-1})$ remains open.

In 2004, Lu and Wu [12] discussed the L^p-mapping properties of T_A under the assumption of that $\Omega \in B_q^{0,v}(S^{n-1})$ for $v = 0,\ 1$. In this paper, we will extend the results of [12] to the weighted cases. Before stating our results, let us first recall some related definitions.

Definition 1.1 (cf. [11]). *A q-block on S^{n-1} is an L^q $(1<q\le\infty)$ function $b(\cdot)$ that satisfies*

(i) $\mathrm{supp}(b) \subset Q$,

(ii) $\|b\|_{L^q(S^{n-1})} \le |Q|^{1/q-1}$,

where $Q = S^{n-1} \cap \{y \in \mathbb{R}^n : |y-\zeta| < \rho$ for some $\zeta \in S^{n-1}$ and $0 < \rho \le 1\}$.

Definition 1.2 (cf. [11]). *For $v > -1$, the block spaces $B_q^{0,v}$ on S^{n-1} are defined by*

$$B_q^{0,v}(S^{n-1}) = \Big\{\Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_s C_s b_s(y'),\ M_q^{0,v}(\{C_s\}) < \infty\Big\},$$

where $\{C_s\}$ is a sequence complex numbers, each b_s is a $q-$block supported in Q_s, and

$$M_q^{0,v}(\{C_s\}) = \sum_s |C_s| \left\{1 + \left(\log^+ \frac{1}{|Q_s|}\right)^{v+1}\right\}.$$

Definition 1.3 (cf. [13]). (i) *A real-valued polynomials $P(x,y)$ is called non-trivial if $P(x,y)$ cannot be written as $P_0(x) + P_1(y)$, where P_0 and P_1 are polynomials define on $\mathbb{R}^n$.*