Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 2 (2024), pp. 191-207

DOI: 10.4208/ata.OA-2023-0027
June 2024

The Existence and Multiplicity of Normalized Solutions for Kirchhoff Equations in Defocusing Case

Lin Xu*

School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, Guangdong 510520, China

Received 5 October 2023; Accepted (in revised version) 25 March 2024

Abstract. In this paper, we study the existence of solutions for Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$

with mass constraint condition

$$S_c:=\left\{u\in H^1\left(\mathbb{R}^3\right):\int_{\mathbb{R}^3}|u|^2dx=c\right\},$$

where a, b, $c>0$, $\mu\in\mathbb{R}$ and $2<q<p<6$. The $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier. For the range of p and q, the Sobolev critical exponent 6 and mass critical exponent $\frac{14}{3}$ are involved which corresponding energy functional is unbounded from below on S_c. We consider the defocusing case, i.e. $\mu<0$ when (p,q) belongs to a certain domain in $\mathbb{R}^2$. We prove the existence and multiplicity of normalized solutions by using constraint minimization, concentration compactness principle and Minimax methods. We partially extend the results that have been studied.

Key Words: Normalized solutions, Kirchhoff-type equation, mixed nonlinearity.

AMS Subject Classifications: 35B08, 35J47, 35P30, 35Q55

1 Introduction and main results

In this paper, we study the existence of solutions with prescribed mass to the following Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

*Corresponding author. *Email address:* xl1244176808@163.com (L. Xu)

where $a,b>0$, $2<q<p<6$. The Eq. (1.1) is closely related to the equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(x,u), \tag{1.2}$$

which is the stationary analog of the equation

$$u_{tt}-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(x,u), \tag{1.3}$$

where $f(x,u)$ is a general nonlinearity. The Eq. (1.3) was initially proposed by Kirchhoff in 1883 as an extension of the classical D'Alembert's wave equations.

After the pioneering work of Lions [1], the Kirchhoff type Eq. (1.3) began to receive much attention and many researchers studied its steadystate model, see [2–6] for more important research progress.

At present, there are two substantially different view points in terms of the frequency λ in Eq. (1.1). One is to regard the frequency λ as a given constant. In this situation, solutions of Eq. (1.1) are critical points of the corresponding action functional on the working space and has been extensively studied see e.g., [21–24] and the references therein. We point out that the existence, multiplicity and concentration of solutions for Eq. (1.1) involving Sobolev subcritical, critical and supercritical exponents have been extensively studied under different assumptions about the nonlinearity term, see [4] and their references therein.

The other one is to regard the λ as an unknown quantity to the Eq. (1.1). In this situation, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier. Nowadays, some physicists are very interested in the solutions satisfying the normalized condition

$$\int_{\mathbb{R}^3}|u|^2dx=c$$

for a priori given c, since the mass admits a clear physical meaning. For example, from a physical point of view, the normalized condition may represent the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. In addition, such solutions can give a better insight of the dynamical properties, like orbital stability or instability and can describe attractive Bose-Einstein condensates. This type of solutions is usually called prescribed mass solutions or normalized solutions in mathematics. In order to study the solution of Eq. (1.1) satisfying the normalized condition $\int_{\mathbb{R}^3}|u|^2dx=c$, it suffices to consider the critical point of the functional

$$E_\mu(u):=\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\frac{\mu}{q}\|u\|_q^q-\frac{1}{p}\|u\|_p^p$$

on

$$S_c=\left\{u\in H^1\left(\mathbb{R}^3\right):\int_{\mathbb{R}^3}|u|^2dx=c\right\},$$