Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 2 (2024), pp. 208-220

DOI: 10.4208/ata.OA-2018-0005
June 2024

$(0,1;0)$–Interpolation on Semi Infinite Interval $(0,\infty)$

Hari Shankar and Pankaj Mathur*

Department of Mathematics and Astronomy, University of Lucknow, Lucknow, India

Received 29 January 2018; Accepted (in revised version) 29 September 2019

Abstract. In this paper, we have studied a Pál type $(0,1;0)$-interpolation when Hermite and Lagrange data are prescribed on the zeros of Laguerre polynomial $(L_n^{(\alpha)})(x)$, $\alpha > -1$ and its derivative $(L_n^{(\alpha)})'(x)$ respectively. Existence, uniqueness and explicit representation of the interpolatory polynomial $R_n(x)$ has been obtained. A qualitative estimate for $R_n(x)$ has also been dealt with.

Key Words: Pál type interpolation, Laguerre polynomials, estimate, zeros.

AMS Subject Classifications: 42A10

1 Introduction

In 1975, L. G. Pál [12] introduced the following interpolation process. Let

$$-\infty < x_{n,n} < \cdots < x_{1,n} < \infty$$

be a system of distinct real points which are zeros of $W_n(x)$, i.e.,

$$W_n(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

The roots $y_{i,n}$ $(i = 1,2,\cdots,n-1)$ of $W_n'(x)$ are interscaled between the roots of $W_n(x)$, i.e.,

$$-\infty < x_{n,n} < y_{n-1,n} < x_{n-1,n} \cdots < y_{1,n} < x_{1,n} < +\infty. \tag{1.1}$$

Pál proved that for given arbitrary numbers $(\alpha_{i,n})_{i=1}^{n}$ and $(\beta_{i,n})_{i=1}^{n-1}$, there exists a unique interpolatory polynomial $R_n(x)$ of degree $2n-1$ satisfying the conditions:

$$R_n(x_{i,n}) = \alpha_{i,n}, \quad i = 1,2,\cdots,n, \quad R_n'(y_{i,n}) = \beta_{i,n-1}{}^{\dagger}, \quad i = 1,2,\cdots,n-1,$$

*Corresponding author. *Email addresses:* harimaurya500@gmail.com (H. Shankar), pankaj_mathur14@yahoo.co.in (P. Mathur)

†For sake of convenience we shall use "i" in place of "i, n".

and an initial condition $R_n(a) = 0$, where a is a given point, different from the nodal points (1.1). Szili [16] was the first to apply this method on infinite interval by taking the mixed nodes of the Hermite polynomial $H_n(x)$ and its derivative $H_n'(x)$. Later I. Joó [6] sharpened his results by improving the estimates of fundamental polynomials. Srivastava and Mathur [15] studied the problem of $(0;0,1)$-interpolation on the mixed zeros of $H_n(x)$ and its derivative. For more results in this direction one is refered to [1,8–11,13,14,19].

Lenard [7] studied a modified Pál type interpolation on Laguerre abscissas and showed that if $(x_i)_{i=1}^n$ and $(x_i^*)_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^k(x)$ and $L_n^{k-1}(x)$, respectively and $x_0 = 0$, then there exists a polynomial $R_m(x)$ of degree 2n+k satisfying the conditions:

$$\begin{aligned} &R_m(x_i) = y_i, \quad R_m'(x_i^*) = y_i', \quad i = 1,2,\cdots,n, \\ &R_m^{(j)}(x_0) = y_0^{(j)}, \quad (j = 0,1,\cdots,k), \end{aligned}$$

where y_i, y_i' and $y_0^{(j)}$ are arbitrary real numbers. She also obtained the explicit representation of the interpolatory polynomial and gave the corresponding quadrature formula.

In this paper, we have considered $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^{n-1}$ to be the zeros of Laguerre Polynomial $L_n^{(\alpha)}(x)$ and its derivative $(L_n^{(\alpha)})'(x)$ respectively, which are interscaled as:

$$0 < x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n < \infty. \tag{1.2}$$

For an arbitrarily given set of real numbers:

$$\{\alpha_k,\ k = 1(1)n;\ \beta_k,\ k = 1(1)n;\ \gamma_k,\ k = 1(1)n-1\}, \tag{1.3}$$

we seek to determine a polynomial $R_n(x)$ of minimal possible degree such that:

$$\begin{cases} R_n(x_k) = \alpha_k, & k = 1,2,\cdots,n, \\ R_n'(x_k) = \beta_k, & k = 1,2,\cdots,n, \\ R_n(y_k) = \gamma_k, & k = 1,2,\cdots,n-1. \end{cases} \tag{1.4}$$

In Section 2 we give some preliminaries. Section 3, is devoted to the existence and explicit representation of the interpolatory polynomials. In Section 4, the estimates of the fundamental polynomials have been obtained and lastly in Section 5 we prove the main theorem of the paper, where the quantitative estimate of the interpolatory polynomial has been dealt with.

2 Preliminaries

Laguerre Polynomial $L_n^{(\alpha)}(x)$ satisfies the differential equation

$$x\left(\frac{d^2y}{dx^2}\right) + (\alpha+1-x)\left(\frac{dy}{dx}\right) + ny = 0, \tag{2.1}$$