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BMO SPACES ASSOCIATED TO GENERALIZED PARABOLIC SECTIONS

Meng Qu
(*Anhui Normal University, China*)
Xinfeng Wu
(*China University of Mining and Technology, China)*

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Abstract. Parabolic sections were introduced by Huang[1] to study the parabolic Monge-Ampère equation. In this note, we introduce the generalized parabolic sections $\mathcal{P}$ and define $\mathrm{BMO}^q_{\mathcal{P}}$ spaces related to these sections. We then establish the John-Nirenberg type inequality and verify that all $\mathrm{BMO}^q_{\mathcal{P}}$ are equivalent for $q \geq 1$.

Key words: $\mathrm{BMO}^q_{\mathcal{P}}$, *generalized parabolic section, John-Nirenberg's inequality*

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1 Introduction

In 1996, Caffarelli and Gutiérrez[1] studied the real variable theory related to the Monge-Ampère equation. They define sections to be a family of convex sets $\mathcal{F} = \{S(x,t) : x \in \mathbf{R}^n \text{ and } t > 0\}$ in $\mathbf{R}^n$ satisfying certain axioms of affine invariance. In term of these sections, they set up a variant of the Calderón-Zygmund decomposition by using the covering lemma and the doubling condition of a Borel measure μ; this decomposition is very important in studying the linearized Monge-Ampère equation[2]. As an application, they defined $\mathrm{BMO}_{\mathcal{F}}(\mathbf{R}^n)$ and showed the John-Nirenberg type inequality. Hardy space $H^1_{\mathcal{F}}(\mathbf{R}^n)$ associated to sections was established by Ding

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and Lin[4]. They also showed that the dual space of $H^1_{\mathcal{F}}(\mathbf{R}^n)$ is just the space $\mathrm{BMO}_{\mathcal{F}}(\mathbf{R}^n)$ defined in [1] and the Monge-Ampère singular integral operator is bounded from $H^1_{\mathcal{F}}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$.

On the other hand, to study the parabolic Monge-Ampère equation, Huang[5] defined the parabolic sections and proved the Besicovitch type covering lemma and Caldrón-Zygmund decomposition associated with these sections.

So a natural question arises: is there a theory of Hardy and BMO type spaces associated to the parabolic sections? In the present note, we want to deal with this problem. More precisely, we introduce the generalized parabolic sections $\mathcal{P}$ and define $\mathrm{BMO}^q_{\mathcal{P}}$ spaces associated to these sections. We then establish the John-Nirenberg type inequality and verify that all $\mathrm{BMO}^q_{\mathcal{P}}$ are equivalent for $q \geq 1$. We remark that Hardy spaces for the generalized parabolic sections have been developed in [6].

Now we give the definition and basic properties of the generalized parabolic sections. Suppose $\varphi(t) : [0,\infty) \to [0,\infty)$ is a monotonic increasing function satisfying

$$\varphi(0) = 0, \qquad \lim_{t\to\infty} \varphi(t) = \infty, \qquad \varphi(2t) \leq C\varphi(t),$$

where C is a constant depending only on φ. Define the generalized parabolic sections by $Q_\varphi(z,r) = S(x,r) \times (t - \varphi(r)/2, t + \varphi(r)/2)$, where S is the (elliptic) sections. Note that if $\varphi(t) = t$, then this definition coincides with that used in [5]. Since we can choose $\varphi(t)$ to be any polynomial in t with nonnegative coefficients and without constant term, thus our definition of parabolic sections are more general. Throughout this paper, we will work for a fixed function φ described as above. Thus we use $Q(z,t)$ to denote the generalized parabolic section without specifying φ. The generalized parabolic sections have the following properties.

(A) There exist positive constants K_1, K_2, K_3, ε_1 and ε_2 with the following property: Given two sections $Q(z_0,r_0)$, $Q(z,r)$ with $r \leq r_0$ and T_p an affine transformation that normalizes $Q(z_0,r_0)$, if

$$Q(z_0,r_0) \cap Q(z,r) \neq \emptyset,$$

then there exists $z' = (x',t') \in B(0,K_3)$ such that

$$\begin{aligned} B\left(x', K_2(\tfrac{r}{r_0})^{\varepsilon_2}\right) \quad &\times \quad \left(t' - \tfrac{\varphi(r)}{2r_0}, t' + \tfrac{\varphi(r)}{2r_0}\right) \subset T_p(Q(z,r)) \\ &\subset \quad B\left(x', K_1(\tfrac{r}{r_0})^{\varepsilon_1}\right) \times \left(t' - \tfrac{\varphi(r)}{2r_0}, t' + \tfrac{\varphi(r)}{2r_0}\right), \end{aligned} \tag{1.1}$$

and

$$T_p z \;=\; (Tx,t') \;\in\; B\left(x', \frac{1}{2}K_2(\frac{r}{r_0})^{\varepsilon_2}\right) \times \{t'\}.$$