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LIOUVILLE PROPERTY FOR A CLASS OF QUASI-HARMONIC SPHERE

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Abstract. In this paper we obtain a Liouville type result for a class of quasi-harmonic spheres with rotational symmetry.

Key words: *Liouville property, quasi-harmonic sphere, rotational symmetry*

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1 Introduction

In [1] Lin and Wang introduced the concept of quasi-harmonic sphere in their study of the heat flow of harmonic maps, and asked whether one can show the existence of such quasi-harmonic spheres. Fan[2] provided the first examples of quasi-harmonic spheres for $N = S^n(3 \leq n \leq 6)$, and Gastel[3] gave more examples with $N = S^n$, for all $n \geq 3$. In a recent paper [4] Ding and Zhao consider the problem on the continuity of quasi-harmonic sphere at ∞, and they show that the non-constant equivariant quasi-harmonic sphere must be discontinuous at infinity. In the present paper we will prove a similar Liouville property for a class of quasi-harmonic spheres with rotational symmetry.

We say u a quasi-harmonic sphere from $\mathbf{R}^n$ to a Riemannian manifold N if it satisfies the following equations

$$\triangle u - \frac{1}{2} x \cdot \nabla u + A(u)(du, du) = 0. \tag{1.1}$$

Note that u is also a harmonic map from $(\mathbf{R}^n, g)$ to N where $g = e^{-\frac{|x|^2}{2(n-2)}} ds_0^2$ and ds_0^2 is the standard Euclidean metric.

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By Nash embedding theorem we can assume N is a Riemannian submanifold of the Euclidean space $\mathbf{R}^k$. We say u is rotational symmetry if it can be represented as

$$u(r,\theta) = (h(r), f(r,h(r))\omega(\theta)), \tag{1.2}$$

where $\omega : S^{n-1} \to S^{m-1}$ is a harmonic map and m is the dimension of N. For simplicity we denote $f(r,h(r))$ by $F(r)$ below.

Our aim in this paper is to prove the following Liouville theorem.

Theorem 1. *If u is rotational symmetry and continuous at the point ∞,* i. e.

$$\lim_{|x|\to\infty} u(x) = y \in N,$$

then u must be a constant map.

2 Proof of the Main Theorem

To prove the theorem we need a simple lemma.

Lemma 1. *Let u be any quasi harmonic sphere from $\mathbf{R}^n$ to N. Then the following equality holds*

$$r^2 \frac{\partial}{\partial r}|u_r|^2 + r(2(n-1) - r^2)|u_r|^2 = \frac{\partial}{\partial r}|u_\theta|^2. \tag{2.3}$$

Proof. As $A(u)(du,du)$ is a norm vector on $\mathbf{N}$, we have

$$< \triangle u, u_r > = \frac{r}{2}|u_r|^2.$$

Using the polar coordinate and the fact $< u_r, u_\theta > = 0$ we can obtain

$$\begin{aligned} \frac{r}{2}|u_r|^2 &= < \triangle u, u_r > \\ &= < u_{rr} + \frac{n-1}{r}u_r + \frac{\triangle_\theta u}{r^2}, u_r > \\ &= \frac{1}{2}\frac{\partial}{\partial r}|u_r|^2 + \frac{n-1}{r}|u_r|^2 - \frac{1}{2r^2}\frac{\partial}{\partial r}|u_\theta|^2, \end{aligned}$$

which implies (2.3).

Now we begin to prove Theorem 1.

The assumption u is rotational symmetry and continuous at ∞ means that in (1.2) there must be

$$\lim_{r\to\infty} F(r) = 0.$$

Noting that $\omega : S^{n-1} \to S^{m-1}$ is harmonic, there exists a constant λ such that

$$|\nabla_\theta \omega| = \lambda. \tag{2.4}$$