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ON SIMULTANEOUS WEAKLY-CHEBYSHEV SUBSPACES

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Abstract. In this paper, we shall introduce and characterize simultaneous quasi-Chebyshev (and weakly-Chebyshev) subspaces of normed spaces with respect to a bounded set S by using elements of the dual space.

Key words: *dual space, best simultaneous approximation, simultaneous quasi-Chebyshev subspace, simultaneous weakly-Chebyshev subspace*
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1 Introduction

Let W be a subspace of the normed space X and $x \in X$. We say that $w_0 \in W$ is a best approximation of x whenever $\|x - w_0\| = d(x,W) = \inf_{w\in W} \|x - w\|$. We denote the set of all best approximations of x by $P_W(x)$. It is known that $P_W(x)$ is a closed, bounded and convex subset for all $x \in X$. A subspace W is called pseudo-Chebyshev (proximinal) if dim $P_W(x) < \infty$ ($P_W(x) \neq \emptyset$) for all $x \in X$[19]. In 2000, Mohebi and Radjavi generalized this notion to quasi-Chebyshev subspaces[9],[10]. Then, Mohebi, Rezapour and Mazaheri generalized the notion of quasi-Chebyshev subspaces to weakly-Chebyshev subspaces[11],[13]. In 2008, Shams, Mazaheri and Vaezpour provided the notion of w-Chebyshev subspaces[17]. On the other hand, the theory of best simultaneous approximation is a generalization of best approximation theory and has been studied by many authors (for example, [1]-[8], [14]- [16], [18], [20]-[22]). But, in these work there is not any characterization about best simultaneous approximation by using the dual space of the normed space. In this paper, we shall introduce and characterize simultaneous quasi-Chebyshev (and weakly-Chebyshev) subspaces of normed spaces with respect to a bounded set S by using elements of the dual space.

Suppose that X is a normed linear space, W a subset of X and S a bounded set in X. We define

$$d(S,W) = \inf_{w\in W} \sup_{s\in S} \|s-w\|.$$

An element $w_0 \in W$ is called a best simultaneous approximation to S from W whenever $d(S,W) = \sup_{s\in S} \|s-w_0\|$. The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$. In the case $S=\{x\}$, $x\in X$, $S_W(S)$ is the set of all best approximations of x in W, $P_W(x)$. Thus, the simultaneous approximation theory is a generalization of best approximation theory in a sense.

A subset W is called S-simultaneous proximinal (or simply, simultaneous proximinal) if $S_W(S)\neq\emptyset$. Also, W is called simultaneous quasi-Chebyshev (simultaneous weakly-Chebyshev) if $S_W(S)$ is nonempty and compact (weakly compact) subset of W. Throughout this paper, we suppose that S is either a bounded or a finite set. Also, we shall use the following Lemma in the sequel which has been proved in [1] and [5].

Lemma 1.1. *Let X be a normed linear space and M a proximinal subspace of X. Then, for each non-empty bounded set S in X we have*

$$d(S,M) = \sup_{s\in S} \inf_{m\in M} \|s-m\|.$$

2 Main Results

Now, we are ready to state and prove our main results. Usually, using finite sets are interesting and give us some ideas. Therefore, in this section first we suppose that S is a finite set. First, we give some elementary results.

Lemma 2.1. *Let X be a normed space, W a subspace of X and $S=\{s_1,\cdots,s_m\}$ a subset of X with linearly independent elements. Suppose that $Y=\operatorname{span} S$. If $Y\cap W=\{0\}$, then there is a bounded linear functional f_0 on X such that $f_0|_W=0$, $f_0|_S=1$ and* $\dfrac{1}{d(S,W)} \leq \|f_0\|$.

Proof. First, consider the subspace

$$Z = \operatorname{span}\,\{W,S\} = \left\{ w - \sum_{i=1}^{m} \alpha_i s_i \;\; \alpha_1,\alpha_2,\cdots,\alpha_m \text{ are scalars and } w\in W \right\}.$$