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TRIGONOMETRIC APPROXIMATION IN REFLEXIVE ORLICZ SPACES

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Abstract. The Lipschitz classes $\mathrm{Lip}(\alpha, M), 0 < \alpha \le 1$ are defined for Orlicz space generated by the Young function M, and the degree of approximation by matrix transforms of $f \in Lip(\alpha, M)$ is estimated by $n^{-\alpha}$.

Key words: *Lipschitz class, matrix transform, modulus of continuity, Nölund transform, Orlicz space*

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1 Introduction and the Main Results

A convex and continuous function $M : [0,\infty) \to [0,\infty)$, for which $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{M(x)}{x} = \infty$$

is called a Young function. The complementary Young function N of M is defined by

$$N(y) := \max\{xy - M(x) : x \ge 0\}$$

for $y \ge 0$.

Let M be a Young function. We denote by $\widetilde{L}_M = \widetilde{L}_M([0,2\pi])$ the set of $2\pi-$periodic measurable functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$\int_0^{2\pi} M(|f(x)|)\,\mathrm{d}x < \infty.$$

The linear span of $\widetilde{L}_M$ is denoted by $L_M = L_M([0,2\pi])$. Equipped with the norm

$$\|f\|_M := \sup\left\{ \int_0^{2\pi} |f(x)g(x)|\,\mathrm{d}x : \int_0^{2\pi} N(|g(x)|)\,\mathrm{d}x \le 1 \right\},$$

where N is the complementary function of M, L_M becomes a Banach space, called the Orlicz space generated by M.

The Orlicz spaces are known as the generalization of the Lebesgue spaces; in special case, the Orlicz space generated by the Young function $M_p(x) = x^p/p, 1 < p < \infty$, is isometrically isomorphic to the Lebesgue space L_p. More general information about Orlicz spaces can be found in [6], [11] and [12].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and let

$$h(t) := \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(tx)}, \qquad t > 0.$$

The numbers α_M and β_M defined by

$$\alpha_M := \lim_{t\to\infty} -\frac{\log h(t)}{\log t}, \qquad \beta_M := \lim_{t\to 0^+} -\frac{\log h(t)}{\log t}$$

are called the lower and upper Boyd indices of the Orlicz space L_M, respectively. It is known that the Boyd indices satisfy

$$0 \le \alpha_M \le \beta_M \le 1$$

and

$$\alpha_N + \beta_M = 1, \qquad \alpha_M + \beta_N = 1.$$

The Orlicz space L_M is reflexive if and only if its Boyd indices are nontrivial, that is $0 < \alpha_M \le \beta_M < 1$ (see, for example [5]).

If $1 \le q < 1/\beta_M \le 1/\alpha_M < p \le \infty$, then $L_p \subset L_M \subset L_q$, where the inclusions being continuous, and hence the relation $L_\infty \subset L_M \subset L_1$ holds. We refer to [1] and [2] for a complete discussion of Boyd indices properties.

The modulus of continuity of the function $f \in L_M$ is defined by

$$\omega(f,\delta)_M = \sup_{0<h\le\delta} \|f(\cdot + h) - f\|_M, \qquad \delta > 0.$$

Let $0 < \alpha \le 1$. The Lipschitz class $\mathrm{Lip}(\alpha, M)$ is defined as

$$\mathrm{Lip}(\alpha, M) = \{f \in L_M : \omega(f,\delta)_M = O(\delta^\alpha), \delta > 0\}.$$

Let $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1.7}$$

Denote by $S_n(f)(x)$, $n = 0, 1, \cdots$ the nth partial sums of the series (1.7) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} u_k(f)(x),$$