

## MEASURES OF WEAK NONCOMPACTNESS AND FIXED POINT THEORY FOR 1-SET WEAKLY CONTRACTIVE OPERATORS ON UNBOUNDED DOMAINS

Shaoyuan Xu

(*Hanshan Normal University, China*)

Afif Ben Amar

(*Université de Gafsa, Cité Universitaire Zarrouk, Tunisie*)

Received July 13, 2010

© Editorial Board of Analysis in Theory & Applications and Springer-Verlag Berlin Heidelberg 2011

**Abstract.** The main purpose of this paper is to prove a collection of new fixed point theorems and existence theorems for the nonlinear operator equation  $F(x) = \alpha x$  ( $\alpha \geq 1$ ) for so-called 1-set weakly contractive operators on unbounded domains in Banach spaces. We also introduce the concept of weakly semi-closed operator at the origin and obtain a series of new fixed point theorems and the existence theorems for the nonlinear operator equation  $F(x) = \alpha x$  ( $\alpha \geq 1$ ) for such class of operators. As consequences, the main results generalize and improve the relevant results, which are obtained by O'Regan and A. Ben Amar and M. Mnif in 1998 and 2009 respectively. In addition, we get the famous fixed point theorems of Leray-Schauder, Altman, Petryshyn and Rothe type in the case of weakly sequentially continuous, 1-set weakly contractive ( $\mu$ -nonexpansive) and weakly semi-closed operators at the origin and their generalizations. The main condition in our results is formulated in terms of axiomatic measures of weak compactness.

**Key words:** *measure of weak noncompactness, weakly condensing and weakly nonexpansive, weakly sequentially continuous, weakly semi-closed at the origin, fixed point theorem*

**AMS (2010) subject classification:** 47H10, 47J05, 47J10

---

\*Supported in part by the Foundation of Hanshan Normal University, China.

## 1 Introduction

The famous fixed point theorems of Leray-Schauder, Altman, Petryshyn and Rothe type for completely continuous operators take an important role in the study of fixed point theory. Since 1-set contractive mappings are a broader class of operators including completely continuous operators, the study of fixed point problems for 1-set contractive operators especially from a closed convex subset into itself has been one of the main objects of research in nonlinear functional analysis and was started by Sadovskii<sup>[23]</sup> Petryshyn<sup>[21,22]</sup>, and Nussbaum<sup>[18]</sup>. Since then, whether a 1-set contractive mapping defined on the closure of bounded open subset of a Banach space has a fixed point, has become an interesting problem<sup>[16,17,26,28]</sup>. For example, in [16] the author defined the fixed point index of 1-set-contractive operators, introduced the concept of semi-closed 1-set-contractive operator and obtained some fixed point theorems of such a class of operators. These studies are mainly based on the potential tool of degree theory in terms of Kuratowski measure of noncompactness. Because the weak topology is the convenient and natural setting to investigate the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned results cannot be easily applied. These equations can be transformed into fixed point problems and nonlinear operator equations involving a broader class of nonlinear operators, in which the operators have the property that the image of any set in a certain sense more weakly compact than the original set itself. The major problem to face is that an infinite dimensional Banach space equipped with its weak topology does not admit open bounded sets. That is, a weakly closed and bounded subset has an empty weak interior and thus coincides with its weak boundary which yields very difficult the verification of the boundary conditions. To this interest, we introduce the concept of weakly semi-closed operator at the origin (see Definition 2.6). The notion of the measure of weak noncompactness was introduced by De Blasi in 1977, see [9]). This index has found applications in operator theory (see [14, 15]) and many existence results for weak solutions of differential and integral equations in Banach spaces (see [8, 20, 25] and other). Recall that weak solutions of the Cauchy problem in reflexive Banach spaces were investigated by Szép<sup>[25]</sup> and weak solutions of nonlinear integral equations in these spaces by O'Regan<sup>[20]</sup>. But, it is not easy to construct some formulas which allow to express the measure of weak noncompactness in a convenient form. This is the reason for introducing the notion of axiomatic measures of weak noncompactness, see [4]. In this paper, we prove a collection of new fixed point theorems and existence theorems for the nonlinear operator equation  $F(x) = \alpha x$  ( $\alpha \geq 1$ ) for so-called 1-set weakly contractive operators on unbounded domains in Banach spaces. We also introduce the concept of weakly semi-closed