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WEIGHTED APPROXIMATION OF r-MONOTONE FUNCTIONS ON THE REAL LINE BY BERNSTEIN OPERATORS

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Abstract. In this paper, we give error estimates for the weighted approximation of r-monotone functions on the real line with Freud weights by Bernstein-type operators.

Key words: *Freud weight, r-monotone function, Bernstein-type operator*

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1 Introduction

For an integer $r \geq 0$, let $C^r(S)$ denote the set of all r-times continuously differentiable functions on S, where $C^0(S) = C(S)$ is the usual set of all continuous functions on S.

Let

$$w(x) = e^{-Q(x)}, \qquad x \in (-\infty, +\infty)$$

be a Freud weight, with the continuous function $Q(x)$ satisfying the following conditions:

(a) $Q \in C^2(0, \infty)$ is a positive even function;

(b) $\lim\limits_{x\to\infty} x\frac{Q''(x)}{Q'(x)} = \gamma > 0$;

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(c) if $\gamma = 1$ or 3, then Q'' is nondecreasing. (see [2, Definition 11.3.1, p.184]).

Evidently, we have the following proposition (see [7, Lemma 1]).

Proposition A. Let the continuous function $Q(x)$ satisfying the conditions (a),(b),(c). Then $\lim_{x\to\infty} Q'(x) = \infty$, and there exist $t_0 > 0$ and $A > 1$ such that

$$\begin{cases} Q'(x) & > & 0, \\ Q''(x) & > & 0, \\ Q'(2x) & \leq & AQ'(x) \end{cases}$$

hold for $x > t_0$.

For a Freud weight $w(x)$, denote by C_w the space of all $f \in C(R)$ such that $\lim_{|x|\to\infty} (wf)(x) = 0$ and equipped with the norm $\|wf\|_{C_w} = \sup_{x\in R} |(wf)(x)|$. We also put

$$\|wf\|_{[c,d]} = \sup_{x\in[c,d]} |(wf)(x)|.$$

For $f \in C_w$ the weighted modulus of smoothness is

$$\begin{aligned} \omega_2(f,t)_w &= \sup_{0<h\leq t} \|w\Delta_h^2 f\|_{[-h^*,h^*]} + \inf_{\ell\in\mathcal{P}_1} \|w(f-\ell)\|_{[t^*,\infty)} \\ &\quad + \inf_{\ell\in\mathcal{P}_1} \|w(f-\ell)\|_{(-\infty,-t^*]}, \end{aligned} \tag{1.1}$$

where h^* and t^* are defined by $hQ'(h^*) = 1$ and $tQ'(t^*) = 1$ respectively (see [2, Definition 11.2.2, p.182]), $\mathcal{P}_n, n \in \mathbf{N}$, is the set of algebraic polynomials of degree at most n, and

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + \frac{rh}{2} - ih\right)$$

is the r-th symmetric difference of f (see [2, p. 7]).

Let the sequence of positive real numbers $\{\lambda_n\}$ be monotone increasing and defined by

$$\lambda_n Q'(\lambda_n) = \sqrt{n}, \qquad n > n_0, \tag{1.2}$$

with n_0 sufficiently large (see [2, p. 7]).It follows from (1.2) that $\lim_{n\to\infty} \frac{\lambda_n}{\sqrt{n}} = 0$.

In the following c, c_1, c_2 denote positive constants which may assume different values in different formulas.

For every $f \in C_w$ let

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_k) \tag{1.3}$$