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SOME RESULTS ON TOPICAL FUNCTIONS AND UPWARD SETS

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Abstract. The purpose of this paper is to introduce and discuss the concept of topical functions on upward sets. We give characterizations of topical functions in terms of upward sets.

Key words: *topical function, upward set, ordered Banach space*

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1 Introduction

If X is a partially ordered vector space X, then the set $X^+ = \{x \in X : \ x \geq 0\}$ is called the positive cone of X, and its members are called positive elements of X.

A partially ordered vector space X is called a vector lattice if for every pair of points x, y in X both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. That is, $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$. In a vector lattice, the positive part, the negative part and the absolute value of an element x are defined by

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and} \quad |x| = x \vee (-x),$$

respectively. Also we have

$$x = x^+ - x^-, \quad |x| = x^+ + x^-, \quad \text{and} \quad |x^+ - y^+| \leq |x - y|.$$

A norm $\|.\|$ on a vector lattice X is said to be a lattice norm, whenever $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$. A normed vector lattice is a vector lattice equipped with a lattice norm. If a normed vector lattice X is complete, then X is referred to a Banach lattice.

Recall that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbf{R}$ such that $x \leq \lambda \mathbf{1}$. Using a strong unit $\mathbf{1}$ we can prove that

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\}, \qquad \forall x \in X$$

is a norm lattice on X. We have also

$$|x| \leq \|x\| \mathbf{1}, \qquad \forall x \in X.$$

Well-know examples of the Banach lattice with strong units are the lattice of all bounded functions defined on a set X and the lattice $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions on a space S with a σ-algebra of measurable sets Σ and a measure μ.

A function $f : X \to \overline{R} = [-\infty, +\infty]$ is called topical if it is increasing $(x \leq y \Longrightarrow f(x) \leq f(y))$ and plus-homogeneous if $f(x + \lambda \mathbf{1}) = f(x) + \lambda$ for all $x \in X$ and all $\lambda \in \mathbf{R}$, and they are studied in [4-5]. The reader may find many applications in applied mathematics (see [3]).

Recall (see [3]) that a subset U of X is said to be upward, if $u \in U$ and $x \in X$ with $u \leq x$, then $x \in U$.

For any subset U of X, we shall denote by int$U$, cl$U$, and bd$U$ the interior, the closure and the boundary of U, respectively. We have

$$N(x, r) := \{y \in X : \ \|x - y\| \leq r\} = \{y \in X : \ x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}.$$

At first we stste the following lemma which is needed in the proof of the main results.

Lemma 1.1[4]**.** *Let $f : X \longrightarrow \bar{R}$ be a topical function. Then the following statements are true:*

(a) *If $x \in X$ and $f(x) = +\infty$ then $f \equiv +\infty$.*

(b) *If $x \in X$ and $f(x) = -\infty$ then $f \equiv -\infty$.*