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ALMOST HOMOMORPHISMS BETWEEN UNITAL C^*-ALGEBRAS: A FIXED POINT APPROACH

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Abstract. Let A,B be two unital C^*-algebras. By using fixed pint methods, we prove that every almost unital almost linear mapping $h : A \longrightarrow B$ which satisfies $h(2^n uy) = h(2^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n = 0,1,2,\cdots$, is a homomorphism. Also, we establish the generalized Hyers–Ulam–Rassias stability of $*-$homomorphisms on unital C^*-algebras.

Key words: *alternative fixed point, Jordan *-homomorphism*

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1 Introduction

A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$". Such a problem was formulated by S.M. Ulam[27] in 1940 and solved in the next year for the Cauchy functional equation by D.H. Hyers[4]. It gave rise to the *stability* *theory* for functional equations. In 1978, Th. M. Rassias[19] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [19] is called the Hyers–Ulam–Rassias stability. Subsequently, various approaches to the problem have been introduced by several authors. For the history and various aspects of this theory we refer the reader to monographs [3, 4, 6, 7, 8] and [10]–[26].

Let A be a unital C^*-algebra with unit e, and B a unital C^*-algebra. Let $U(A)$ be the set of unitary elements in A, $A_{sa} := \{x \in A | x = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa} | \|v\| = 1, v \in Inv(A)\}$.

A unital C^*-algebra is of real rank zero, if the set of invertible self–adjoint elements is dense in the set of self–adjoint elements (see [1]).

Recently, C. Park, D.-H. Boo and J.-S. An[17] investigated almost homomorphisms between unital C^*-algebras.

In this paper, we will adopt the fixed point alternative of Cădariu and Radu to investigate the $*-$homomorphisms, and the generalized Hyers–Ulam–Rassias stability of $*-$homomorphisms on unital C^*-algebras associated with the Jensen–type functional equation

$$f(\frac{x+y}{2}) + f(\frac{x-y}{2}) = f(x).$$

In section two, we prove that every almost unital almost linear mapping $h : A \longrightarrow B$ is a homomorphism when $h(2^n uy) = h(2^n u)h(y)$ holds for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, ...$, and that for a unital C^*-algebra A of real rank zero (see [1]), every almost unital almost linear continuous mapping $h : A \longrightarrow B$ is a homomorphism when $h(2^n uy) = h(2^n u)h(y)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \cdots$.

In section three, we establish the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms on unital C^*-algebras.

Throughout this paper assume that A, B are two C^*-algebras. For a given mapping $f : A \to B$, we define

$$\Delta_\mu f(x,y) = \mu f(\frac{x+y}{2}) + \mu f(\frac{x-y}{2}) - f(\mu x)$$

for all $\mu \in \mathbf{T} := \{z \in \mathbf{C}, |z| \leq 1\}$ and all $x, y \in A$. We denote the algebric center of algebra A by $Z(A)$.

2 $*$-Homomorphisms

Before proceeding to the main results, we will state the following theorem (see [19, 27]).