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SOME INTEGRAL INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then it is recently proved by Rather [*Jour. Ineq. Pure and Appl. Math.*, 9 (2008), Issue 4, Art. 103] that for every $\gamma > 0$ and every real or complex number α with $|\alpha| \geq 1$,

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma} \leq n(|\alpha|+1)C_\gamma \left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma},$$

$$C_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\beta}|^\gamma \mathrm{d}\beta\right\}^{-1/\gamma},$$

where $D_\alpha P(z)$ denotes the polar derivative of $P(z)$ with respect to α. In this paper we prove a result which not only provides a refinement of the above inequality but also gives a result of Aziz and Dawood [J. Approx. Theory, 54 (1988), 306-313] as a special case.

Key words: *polar derivative, polynomial, Zygmund inequality, zeros*

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1 Introduction and Statement of Results

Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree atmost n and $P'(z)$ its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \tag{1.1}$$

and for every $\gamma \geq 1$,

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^\gamma |rmd\theta\right\}^{1/\gamma} \leq n\left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}. \tag{1.2}$$

The inequality (1.1) is a classical result of Bernstein[11] (see also [14]), whereas the inequality (1.2) is due to Zygmund[15], who proved it for all trigonometric polynomials of degree n and not only for those of the form $P(e^{i\theta})$. Arestov[1] proved that (1.2) remains true for $0<\gamma<1$ as well. If we let $\gamma\to\infty$ in the inequality (1.2), we get (1.1).

The above two inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$. In fact, if $P(z)\neq 0$ in $|z|<1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{2}\max_{|z|=1}|P(z)| \tag{1.3}$$

and

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}\leq nB_\gamma\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}, \tag{1.4}$$

where

$$B_\gamma=\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^\gamma \mathrm{d}\alpha\right\}^{-1/\gamma}.$$

The inequality (1.3) is conjectured by Erdös and later verified by Lax[9], whereas the inequality (1.4) is proved by De-Bruijn [7] for $\gamma\geq 1$. Further, Rahman and Schmeisser[12] have shown that (1.4) holds for $0<\gamma<1$ also. If we let $\gamma\to\infty$ in the inequality (1.4), we get (1.3).

The inequality (1.3) is further improved by Aziz and Dawood[4] by proving that if $P(z)\neq 0$ in $|z|<1$, then

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{2}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\right\}. \tag{1.5}$$

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to a complex number α. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z).$$

Aziz[3] extended the inequality (1.3) to the polar derivatives and proved that if $P(z)$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<1$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)|\leq\frac{n}{2}(|\alpha|+1)\max_{|z|=1}|P(z)|. \tag{1.6}$$