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ON DOUBLE SINE AND COSINE TRANSFORMS, LIPSCHITZ AND ZYGMUND CLASSES

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Abstract. We consider complex-valued functions $f \in L^1(\mathbf{R}^2_+)$, where $\mathbf{R}_+ := [0, \infty)$, and prove sufficient conditions under which the double sine Fourier transform $\hat{f}_{ss}$ and the double cosine Fourier transform $\hat{f}_{cc}$ belong to one of the two-dimensional Lipschitz classes $\mathrm{Lip}(\alpha, \beta)$ for some $0 < \alpha,\ \beta \le 1$; or to one of the Zygmund classes $\mathrm{Zyg}(\alpha, \beta)$ for some $0 < \alpha,\ \beta \le 2$. These sufficient conditions are best possible in the sense that they are also necessary for nonnegative-valued functions $f \in L^1(\mathbf{R}^2_+)$.

Key words: *double sine and cosine Fourier transform, Lipschitz class* $\mathrm{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \le 1$, *Zygmund class* $\mathrm{Zyg}(\alpha, \beta)$, $0 < \alpha$, $\beta \le 2$.

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1 Known Results: Single Sine and Cosine Transforms

We consider complex-valued functions $f : \mathbf{R}_+ \to \mathbf{C}$ that are integrable in Lebesgue sense over $\mathbf{R}_+ := [0, \infty)$, in symbol: $f \in L^1(\mathbf{R}_+)$. We recall that the sine (Fourier) transform of f is defined by

$$\hat{f}_s(u) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin ux \mathrm{d}x,$$

while the cosine (Fourier) transform of f is defined by

$$\hat{f}_c(u) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos ux \mathrm{d}x, \qquad u \in \mathbf{R}.$$

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Both $\hat{f}_s$ and $\hat{f}_c$ are uniformly continuous on $\mathbf{R}$ and vanish at infinity. For details, we refer to [6, Ch. 1].

In the cases when we do not distinguish between $\hat{f}_s$ and $\hat{f}_c$, we simply use the notation $\hat{f}$. We recall that $\hat{f}$ is said to satisfy the Lipschitz condition of order $\alpha > 0$, in symbol: $\hat{f} \in \text{Lip}(\alpha)$, if

$$|\hat{f}(u+h) - \hat{f}(u)| \le Ch^{\alpha} \quad \text{for all} \quad u \in \mathbf{R} \quad \text{and} \quad h > 0,$$

where the constant C does not depend on u and h. Furthermore, $\hat{f}$ is said to satisfy the Zygmund condition of order $\alpha > 0$, in symbol: $\hat{f} \in \text{Zyg}(\alpha)$, if

$$|\hat{f}(u+h) - 2\hat{f}(u) + \hat{f}(u-h)| \le Ch^{\alpha} \quad \text{for all} \quad u \in \mathbf{R} \quad \text{and} \quad h > 0,$$

where the constant C does not depend on u and h.

It is well known (see, e.g., [1, Ch. 2] or [7, Ch. 2, §3] that if $\hat{f} \in \text{Lip}(\alpha)$ for some $\alpha > 1$, or if $\hat{f} \in \text{Zyg}(\alpha)$ for some $\alpha > 2$, then $\hat{f} \equiv 0$.

The following four theorems were proved in [4] by the second named author of the present paper.

Theorem A. (i) *Let* $f : \mathbf{R}_+ \to \mathbf{C}$ *be such that* $f \in L^1_{\text{loc}}(\mathbf{R}_+)$. *If for some* $0 < \alpha \le 1$,

$$\int_0^s x|f(x)| = O(s^{1-\alpha}) \quad \textit{for all} \quad s > 0, \tag{1.1}$$

then $f \in L^1(\mathbf{R}_+)$ *and* $\hat{f}_s \in \text{Lip}(\alpha)$.

(ii) *Let* $f : \mathbf{R}_+ \to \mathbf{R}_+$ *be such that* $f \in L^1(\mathbf{R}_+)$. *If* $\hat{f}_s \in \text{Lip}(\alpha)$ *for some* $0 < \alpha \le 1$, *then* (1.1) *holds.*

Theorem B. *In case* $0 < \alpha < 1$, *Theorem A remains valid when* $\hat{f}_s$ *is replaced by* $\hat{f}_c$.

Theorem C. (i) *Let* $f : \mathbf{R}_+ \to \mathbf{C}$ *be such that* $f \in L^1_{\text{loc}}(\mathbf{R}_+)$. *If for some* $0 < \alpha \le 2$,

$$\int_0^s x^2|f(x)| = O(s^{2-\alpha}) \quad \textit{for all} \quad s > 0, \tag{1.2}$$

then $f \in L^1(\mathbf{R}_+)$ *and* $\hat{f}_c \in \text{Zyg}(\alpha)$.

(ii) *Let* $f : \mathbf{R}_+ \to \mathbf{R}_+$ *be such that* $f \in L^1(\mathbf{R}_+)$. *If* $\hat{f}_c \in \text{Zyg}(\alpha)$ *for some* $0 < \alpha \le 2$, *then* (1.2) *holds.*

Theorem D. *In case* $0 < \alpha < 2$, *Theorem C remains valid when* $\hat{f}_c$ *is replaced by* $\hat{f}_s$.

Our goal in this paper is to extend these results from single to double sine and cosine transform.