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HARMONIC MAPPINGS OF THE HEXAGASKET TO THE CIRCLE

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Abstract. Harmonic mappings from the hexagasket to the circle are described in terms of boundary values and topological data. Explicit formulas are also given for the energy of the mapping. We have generalized the results in [10].

Key words: *hexagasket, harmonic mapping, self-similar Dirichlet form*

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1 Introduction

Whenever there is a theory of harmonic functions on a space X, there should be a theory of harmonic mappings from X to Y, where the target space Y is any Riemannian manifold(see [1], [2] for more details). In this paper we take Y to be the circle, and we want to show that Strichartz's method in [10] holds for the hexagasket. The hexagasket provides us one kind of possibility to show the method holds for all $n-$gasket if we observe the fact that for the general $n-$gasket we can find a boundary set is V_0 only consisting of 3 vertices(see [9] and Section 4.1 in [11]) if n is not a multiple of 4.

The hexagasket[6],[9],[11] is generated by the i.f.s. consisting of 6 mappings in the plane, $F_i(x) = \frac{1}{3}(x-p_i)+p_i, i = 1,2,3,4,5,6$, where $p_1,\cdots,p_6$ are vertices of a regular hexagon. The usual boundary set $V_0 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. But in this paper we take a smaller boundary $V_0 = \{p_1, p_3, p_5\}$, and the hexagasket is also an affine nested fractal(see [7]).

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We approximate the hexagasket K by a sequence of graphs $\Gamma_0, \Gamma_1, \cdots$ with vertices $V_0 \subseteq V_1 \subseteq V_2 \cdots$, and $V_{k+1} = \cup_{j=1}^{6} F_j V_k$. The edge relation for Γ_m, denoted $x \sim_m y$, for $x, y \in V_m$ and $x \neq y$, is defined by the existence of a word $w = (w_1, \cdots, w_m)$ with length $|w| = m$ such that $x, y \in F_w K$, where $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$. The simple energy form on Γ_m is

$$E_m(u,v) = \sum_{x \sim_m y} (u(x) - u(y))(v(x) - v(y)), \tag{1.1}$$

and the renormalization energy ε_m is given by

$$\varepsilon_m(u,v) = (\frac{7}{3})^m E_m(u,v), \tag{1.2}$$

where u and v denote continuous functions on K and, by abuse of notation, their restriction to V_m.

We regard V_0 as the boundary of each graph V_m, and also of K. A function h on V_m (for $m \geq 1$) is called graph harmonic if it satisfies

$$h(x) = \frac{1}{n} \sum_{y \sim_m x} h(y), \quad \text{for} \quad \sharp\{y : y \sim_m x\} = n = 2 \text{ or } 4, \tag{1.3}$$

for all non-boundary point x. It is easy to see this is equivalent to the property that h minimizes the energy $E_m(u,u)$ among all functions u with the same boundary values.

The following proposition summaries the basic results(from $[3], [4], [5], [6], [8], [11]$) concerning the Dirichlet form and harmonic functions on K, and justifies the choice of renormalization factor r in (1.2):

Proposition 1.1. (i) For any continuous function u on K, the sequence $\varepsilon_m(u,u)$ is monotone increasing, so

$$\varepsilon(u,u) = \lim_{m \to \infty} \varepsilon_m(u,u) \tag{1.4}$$

is well-defined in $[0, \infty]$, and $\varepsilon(u,u) = 0$ if and only if u is a constant.

Denote by $dom(\varepsilon)$ the set of continuous functions for which $\varepsilon(u,u) < \infty$. Then $dom(\varepsilon)$ modulo constants is a Hilbert space with the inner product

$$\varepsilon(u,v) = \lim_{m \to \infty} \varepsilon_m(u,v). \tag{1.5}$$

(ii) A function h is called harmonic on K if it minimizes the energy $\varepsilon(u,u)$ among functions with the same boundary values. Then h is harmonic if and only if its restriction to the every V_m is graph harmonic.

For a harmonic function h, $\varepsilon_m(h,h) = \varepsilon(h,h)$ for every m.

The space of harmonic functions is 3-dimensional, with each harmonic function determined uniquely from its boundary by means of the following harmonic algorithm: if the values of h on V_m are known, and the values $h(x)$ for $x \in V_{m+1} \setminus V_m$ is desired, find w with length $|w| = m$, such that $x \in F_w K$, and set

$$h(x) = D_w \rho(x). \tag{1.6}$$