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A THEORETICAL FRAMEWORK FOR THE CALCULATION OF HAUSDORFF MEASURE — SELF-SIMILAR SET SATISFYING OSC

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Abstract. A theoretical framework for the calculation of Hausdorff measure of self-similar sets satisfying OSC has been established.

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1 Introduction and Nan Results

It is well known that to calculate the Hausdorff measure of fractal sets is very difficult, even for simple sets, such as self-similar sets satisfying the open set condition(OSC,including SSC) and so there are few concrete results about computation of Hausdorff measure, unless the Hausdorff dimension is not larger than 1. Some authors have investigated the estimation and calculation and got some upper and lower bounds of the Hausdorff measure for self-similar sets satisfying OSC(see [6, 7, 8, 10, 11, 12, 14, 19]). A natural question is how to get the accurate value of them? In this paper we only discuss the case of self-similar sets satisfying OSC and our purpose is to establish a uniform theoretical framework for the calculation of Hausdorff measure for such fractal sets. Let $E \subset \mathbf{R}^n$ be a self-similar set satisfying OSC with $s = \dim_H E$. We have proved $H^s(E \cap U) \leq |U|^s$ (which will be called the measure-diameter's inequality) [12] for any $U \subset \mathbf{R}^n$. This inequality plays an important role for calculating the Hausdorff measure of the self-similar set satisfying OSC. The calcutation of the Hausdorff measure of E will be transformed to look for a solution U with $0 < |U|$ such that the equality holds in the above inequality, that is, $H^s(E \cap U) = |U|^s$. In this paper, we prove such a set U exists. Here, our proof is on existence behavior but not constructive and so in order to count genuinely the Hausdorff measure of E, we also must determine wholly U, including its diameter, geometric shape and location. We call this result the realization theorem with respect to the upper convex density 1. The upper convex density, introduced by Falconer [1], is an important notion giving rise to a series of new problems and opening a gate to understand deeply the structure of self-similar set and so far we work on it only a little. Our main results of this paper are as follows (for the definitions, terminologies and notations, see the next paragraph).

In this paper, we always let $E \subset \mathbf{R}^n (n > 0)$ be a self-similar set satisfying OSC and denote by $s = \dim_H E$ its Hausdorff dimension and $H^s(E)$ its $s-$dimensional Hausdorff measure.

Realization theorem. *There exists a set $U \subset \mathbf{R}^n$ with $0 < |U|$ such that*

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = 1.$$

Corollary 1. $H^s(E) = |E \cap U|^s / \sum_{k>0} b_k$, *where $|U| > 0$ satisfies*

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = 1$$

and each b_k depends on U.

Corollary 2. *There exists an almost everywhere best covering of E, $\alpha = \{U_i : i > 0\}$, such that*

$$H^s(E) = \sum_{i>0} |U_i|^s.$$

The proof of these results will be given later. Some discussions are given at the end of this paper.

2 Basic Concepts and Upper Convex Density

For some basic definitions and notations in Fractal Geometry, we refer to [1, 2, 3].

Denote by d the usual metric of $\mathbf{R}^n (n > 0)$. Let $D \subset \mathbf{R}^n$ be a bounded closed region and E a self-similar set yielded by $m (m > 0)$ (linear) similarities $S_j : D \to D$ with ratios $0 < c_j < 1, j = 1, 2, \ldots, m$, that is, $|S_j(x) - S_j(y)| = c_j |x - y|, \quad \forall\, x, y \in D, \; j = 1, 2, \cdots, m$ and E satisfies $E = \bigcup_j S_j(E)$. We say that E satisfies the open set condition (OSC) if there is a non-empty bounded open set $V \subset \mathbf{R}^n$ such that $\bigcup_j S_j(V) \subset V$ and

$$S_i(V) \cap S_j(V) = \emptyset, \; 1 \le i < j \le m.$$