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POLYNOMIALLY BOUNDED COSINE FUNCTIONS

Dingbang Cang
(*North China Institute of Science and Technology, China*)
Xiaoqiu Song
(*China University of Mining and Technology, China*)
Chen Cang
(*North China Institute of Science and Technology, China)*

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Abstract. We characterize polynomial growth of cosine functions in terms of the resolvent of its generator and give a necessary and sufficient condition for a cosine function with an infinitesimal generator which is polynomially bounded.

Key words: *cosine functions, resolvent, polynomially bounded*

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1 Introduction

It is well known that the semigroup theory is a useful tool to deal with the first order Cauchy problems. As an important component of semigroup theory, cosine functions play a similar role for the second order Cauchy problem. Since M.Sova introduces the concept of cosine function in 1966, many mathematicians have studied in this field, and many valuable results have been obtained (see [1-4]).

A classical problem in semigroup theory is to characterize the boundedness of a strongly continuous semigroup. Recently,(see [5-6])bounded and polynomially bounded semigroups and groups have been characterized by using only the first and the second power of resolvent of the generator. In this paper we characterize the polynomial growth of cosine functions in terms of

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the resolvent of its generator and give a necessary and sufficient condition for a cosine function with an infinitesimal generator which is polynomially bounded.

Definition 1.1. A strongly continuous family $\{T(t)\}_{t\geq 0}$ is called a cosine function, if $\{T(t)\}_{t\geq 0}$ satisfies $T(0)=I$ and $2T(S)T(t)=T(S+T)+T(S-T)$.

Definition 1.2. Assume that A is closed,$\lambda^2\in\rho(A)$ and the resolvent of A satisfies

$$R(\lambda^2,A)=\lambda^{-1}\int_a^b e^{-\lambda t}T(t)\mathrm{d}x$$

then A is called the generator of $\{T(t)\}_{t\geq 0}$.

We denote by $s_0(A):=\inf\{a\in R: R(\lambda^2,A)$ that is bouned on $\{\mathrm{Re}\lambda>a\}\}$ the pseudo-spectral bound of A.

Definition 1.3. A strongly continuous family $\{T(t)\}_{t\geq 0}$ is called polynomially bounded if $\|T(t)\|\leq C(1+t^d)$ for some constant $C,d\geq 0$ and all $t\geq 0$.

In this paper we assume the following conditions hold:

(1) $\int_{-\infty}^{\infty}\|(a+is)R((a+is)^2,A)x\|^p\mathrm{d}s<\infty$, for all $x\in X$,

(2) $\int_{-\infty}^{\infty}\|(a+is)R((a+is)^2,A')y\|^q\mathrm{d}s<\infty$, for all $y\in X'$.

where $a,b>s_0(A), 1<p,q<\infty,\dfrac{1}{p}+\dfrac{1}{q}=1$.

Definition 1.4. A Banach space is called of Fourier type p if the Fourier transform extends to a bounded linear operator from $L^p(R,X)$ to $L^q(R,X')$, where

$$\frac{1}{p}+\frac{1}{q}=1.$$

2 Characterization of Polyniomail Growth

Lemma 2.1. *Let a be densely defined on a Banach space X, then for every* $a>s_0(A)$ *and* $x\in X$, $\lambda R(\lambda^2,A)x\to 0,|\lambda|\to\infty$, $\mathrm{Re}\lambda\geq a$.

Proof. Let $a>s_0(A)$.Then there exists a constant $M>0$ such that $\|R(\lambda^2,A)\|\leq M$ for all $Re\lambda\geq a$. Let now $x\in X$ and $\mathrm{Re}\lambda\geq a$, then

$$\|\lambda R(\lambda^2,A)x\|=\frac{1}{|\lambda|}\|x+R(\lambda^2,A)Ax\|\leq\frac{1}{|\lambda|}(\|x\|+M\|Ax\|)$$

and therefore we have $\lambda R(\lambda^2,A)x\to 0,|\lambda|\to\infty,\mathrm{Re}\lambda\geq a$ for all $x\in D(A)$.Since $D(A)$ is dense in X and the resolvent of A is uniformly bouned on $\mathrm{Re}\lambda\geq a$,this is true for all $x\in X$.

Theorem 2.1. *Let a densely defined and closed operator A be the generator of a cosine function* $\{T(t)\}_{t\geq 0}$. *It satisfies the conditions (1) and (2).Assume that* $\mathrm{Re}\lambda>0$ *is contained in the resolvent set of A and there exist* $a_0>0$ *and* $M>0$ *such that the following conditions hold:*