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COMMON FIXED POINTS WITH APPLICATIONS TO BEST SIMULTANEOUS APPROXIMATIONS

Sumit Chandok and T. D. Narang
(*Guru Nanak Dev University, India)*

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Abstract. For a subset K of a metric space (X,d) and $x\in X$,

$$P_K(x)=\left\{y\in K: d(x,y)=d(x,K)\equiv\inf\{d(x,k): k\in K\}\right\}$$

is called the set of best K-approximant to x. An element $g_\circ\in K$ is said to be a best simultaneous approximation of the pair $y_1,y_2\in X$ if

$$\max\left\{d(y_1,g_\circ),d(y_2,g_\circ)\}=\inf_{g\in K}\max\{d(y_1,g),d(y_2,g)\right\}.$$

In this paper, some results on the existence of common fixed points for Banach operator pairs in the framework of convex metric spaces have been proved. For self mappings T and S on K, results are proved on both T- and S- invariant points for a set of best simultaneous approximation. Some results on best K-approximant are also deduced. The results proved generalize and extend some results of I. Beg and M. Abbas[1], S. Chandok and T.D. Narang[2], T.D. Narang and S. Chandok[11], S.A. Sahab, M.S. Khan and S. Sessa[14], P. Vijayaraju[20] and P. Vijayaraju and M. Marudai[21].

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1 Introduction

Let (X,d) be a metric space. A mapping $W: X\times X\times[0,1]\to X$ is said to be (s.t.b.) a **convex structure** on X if for all $x,y\in X$ and $\lambda\in[0,1]$

$$d(u,W(x,y,\lambda))\le\lambda d(u,x)+(1-\lambda)d(u,y)$$

holds for all $u\in X$. The metric space (X,d) together with a convex structure is called a **convex metric space** [19].

A convex metric space (X,d) is said to satisfy **Property (I)**[7] if for all $x,y,p \in X$ and $\lambda \in [0,1]$,

$$d(W(x,p,\lambda),W(y,p,\lambda)) \leq \lambda d(x,y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [19]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X,d) is s.t.b. **convex**[19] if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$. A set K is said to be p**-starshaped** (see [8]) where $p \in K$, provided $W(x,p,\lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$ i.e. the segment

$$[p,x] = \{W(x,p,\lambda) : 0 \leq \lambda \leq 1\}$$

joining p to x is contained in K for all $x \in K$. K is said to be **starshaped** if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A self map T on a metric space (X,d) is s.t.b.

i) **nonexpansive** if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$;

ii) **contraction** if there exists an α, $0 \leq \alpha < 1$ such that $d(Tx,Ty) \leq \alpha d(x,y)$ for all $x,y \in X$.

For a nonempty subset K of a metric space (X,d), a mapping $T : K \to K$ is s.t.b.

i) **demicompact** if every bounded sequence $< x_n >$ in K satisfying $d(x_n,Tx_n) \to 0$ has a convergent subsequence;

ii) **asymptotically** nonexpansive [6] if there exists a sequence $\{k_n\}$ of real numbers in $[1,\infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x),T^n(y)) \leq k_n d(x,y)$, for all $x,y \in K$.

Let $T,S : K \to K$. Then T is s.t.b.

i) S**-asymptotically** nonexpansive if there exists a sequence $\{k_n\}$ of real numbers in $[1,\infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x),T^n(y)) \leq k_n d(Sx,Sy)$, for all $x,y \in K$;

ii) **uniformly asymptotically regular** on K if for each $\varepsilon > 0$ there exists a positive integer N such that $d(T^n(x),T^n(y)) < \varepsilon$ for all $n \geq N$ and for all $x,y \in K$.

A point $x \in K$ is a **common fixed (coincidence) point** of S and T if $x = Sx = Tx$ ($Sx = Tx$). The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S,T)$ (respectively, $C(S,T)$).

The mappings $T,S : K \to K$ are s.t.b. **commuting** on K if $STx = TSx$ for all $x \in K$; R**-weakly commuting**[13] on K if there exists $R > 0$ such that

$$d(TSx,STx) \leq Rd(Tx,Sx)$$

for all $x \in K$; **compatible**[9] if $\lim d(TSx_n,STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M; **weakly compatible**[10] if S and T commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$.