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THE HAUSDORFF MEASURE OF SIERPINSKI CARPETS BASING ON REGULAR PENTAGON*

Chaoyi Zeng, Dehui Yuan
(*Hanshan Normal University, China*)
and
Shaoyuan Xu**
(*Gannan Normal University, China)*

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Abstract. In this paper, we address the problem of exact computation of the Hausdorff measure of a class of Sierpinski carpets E− the self-similar sets generating in a unit regular pentagon on the plane. Under some conditions, we show the natural covering is the best one, and the Hausdorff measures of those sets are euqal to $|E|^s$, where $s = \dim_H E$.

Key words: *Sierpinski carpet, Hausdorff measure, upper convex density*

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1 Introduction

The Hausdorff measure and dimension are the most important concepts in fractal geometry, and their computation is very difficult. Recently, in order to study deeply the Hausdorff measure, the reference [1] gave the notions “best covering" and “natural covering", and posed eight open problems and six conjectures on Hausdorff measure. Using the notion upper convex density of a class of self-similar sets, the reference [2] studied a class of self-similar sets-generating in a unit square on the plane, proved that the natural covering is the best one and the Hausdorff measures of those sets are euqal to $\sqrt{2}^s$. In this paper, we address the problem of the exact computation

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Corresponding author.

of the Hausdorff measure of a class self similar sets-generating in a unit regular pentagon on the plane.

For convenience, we first present some notions that will be used in the rest part of the paper.

Definition 1. Suppose $E \subset \mathbf{R}^2$, $s \in \mathbf{R}$, $s \geq 0$ and $\delta > 0$, the Hausdorff measure of the set E is defined as

$$H^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : |U_i| \leqslant \delta, E \subset \bigcup U_i \right\}$$

where $\{U_i\}_{i=1}^{\infty}$ is arbitrary covering of the set E; and the Hausdorff dimension of E (denoted by $\dim_H E$) is defined as

$$\dim_H E = \sup\{s : H^s(E) = \infty\} = \inf\{s : H^s(E) = 0\}.$$

Definition 2. Let $\delta > 0$, $s \geq 0$, $E \subset R^2$, $x \in E$. Moreover, for a convex set U_x containing x, the upper convex density of E at x is defined as

$$\bar{D}_C^s(E,x) = \lim_{\delta \to 0} \sup_{0<|U_x|<\delta} \left\{ \frac{H^s(E \cap U_x)}{|U_x|^s} \right\}.$$

The properties of the upper convex density are discussed in the reference [5].

Definition 3. (See Fig. 1) Let E_0 be an unit regular pentagon $A_1A_2A_3A_4A_5$ on the plane R^2, E be the attractor generated by the iterated function system (IFS) $\{f_i | i = 1,2,3,4,5\}$, where

$$f_i(x) = \lambda_i x + b_i, 0 < \lambda_i < 1, i = 1,2,3,4,5, x = (x_1, x_2) \in E_0$$
$$b_1 = ((1-\lambda_1)\sin 18^\circ, 0)$$
$$b_2 = ((1-\lambda_2)(\sin 18^\circ + 1), 0)$$
$$b_3 = ((1-\lambda_3)(2\sin 18^\circ + 1), (1-\lambda_3)\cos 18^\circ)$$
$$b_4 = ((1-\lambda_4)\cos 18^\circ, (1-\lambda_4)(\sin 18^\circ + \cos 18^\circ))$$
$$b_5 = (0, (1-\lambda_5)\cos 18^\circ)$$

Then the self-similar set E is called a Sierpinski carpet generating in a unit regular pentagon, where $s = \dim_H E$ satisfies $\sum_{i=1}^{5} \lambda_i^s = 1$.

2 Two Lemmas

In this section, we present two lemmas which will be used in the proof of the main result of this paper.