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A REMARK ON CERTAIN DIFFERENTIAL INEQUALITIES INVOLVING p-VALENT FUNCTIONS

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Abstract. In the present paper, we study certain differential inequalities involving p-valent functions and obtain sufficient conditions for uniformly p-valent starlikeness and uniformly p-valent convexity. We also offer a correct version of some known criteria for uniformly p-valent starlike and uniformly p-valent convex functions.

Key words: *p-valent function, uniformly starlike function, uniformly convex function, uniformly close-to-convex function*

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k, \qquad p\in\mathbf{N}=\{1,2,3,\cdots\},$$

which are analytic and p-valent in the open unit disk $\mathbf{E}=\{z\in\mathbf{C}:|z|<1\}$. A function $f\in\mathcal{A}_p$ is said to be uniformly p-valent starlike in $\mathbf{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\left|\frac{zf'(z)}{f(z)}-p\right|, \qquad z\in\mathbf{E}. \tag{1.1}$$

We denote by US_p^*, the class of uniformly p-valent starlike functions. A function $f\in\mathcal{A}_p$ is said to be uniformly p-valent convex in $\mathbf{E}$ if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\left|1+\frac{zf''(z)}{f'(z)}-p\right|, \qquad z\in\mathbf{E}.$$

Let UC_p denote the class of uniformly p-valent convex functions. A function $f\in\mathcal{A}_p$ is said to be uniformly p-valent close-to-convex in $\mathbf{E}$ if

$$\Re\left(\frac{zf'(z)}{g(z)}\right)>\left|\frac{zf'(z)}{g(z)}-p\right|, \qquad z\in\mathbf{E}, \tag{1.2}$$

for some $g \in US_p^*$. Let UCC_p denote the class of all such functions. Note that the function $g(z) \equiv z^p \in \mathcal{A}_p$ and satisfies the condition (1.1). Therefore, when we select $g(z) \equiv z^p$, in condition (1.2), it reduces to

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \left|\frac{f'(z)}{z^{p-1}} - p\right|, \qquad z \in \mathbf{E}. \tag{1.3}$$

Hence, a function $f \in \mathcal{A}_p$ is uniformly p-valent close-to-convex in $\mathbf{E}$ if it satisfies the condition (1.3).

In 1991, Goodman[2] introduced the concept of uniformly starlike and uniformly convex functions. He defined uniformly starlike and uniformly convex functions as functions $f \in \mathcal{A}$ with the geometric property that the image of every circular arc contained in $\mathbf{E}$, with center $\zeta \in \mathbf{E}$, is starlike with respect to $f(\zeta)$ and convex respectively.

In 1993 Ronning[4] studied the class of uniformly convex functions and obtained an interesting criterion for $f \in \mathcal{A}$ to be uniformly convex in $\mathbf{E}$. He proved that a function $f \in \mathcal{A}$ is uniformly convex if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \qquad z \in \mathbf{E}.$$

For analytic function f and analytic univalent function g in $\mathbf{E}$, we say that f is subordinate to g in $\mathbf{E}$ and write as $f \prec g$ if $f(0) = g(0)$ and $f(\mathbf{E}) \subset g(\mathbf{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \ \Phi(p(0), 0; 0) = h(0). \tag{1.4}$$

A univalent function q is called a dominant of the differential subordination (1.4) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.4). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.4) is said to be the best dominant of (1.4).

Define the parabolic domain Ω and the circular domain O as follows:

$$\Omega = \left\{ u + iv : u > \sqrt{(u-p)^2 + v^2} \right\}$$

and

$$O = \left\{ u + iv : \sqrt{(u-p)^2 + v^2} < \frac{p}{2} \right\}.$$

Obviously $O \subset \Omega$. In 2008, Al-Kharsani and Al-Hajiry[1] proved the following results for uniformly p-valent starlikeness and convexity.

Theorem 1.1. Let $f \in \mathcal{A}_p$ satisfy the following inequality

$$\Re\left(\frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p}\right) < 1 + \frac{2}{3p}, \tag{1.5}$$

then f is uniformly p-valent starlike in $\mathbb{E}$.