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WEIGHTED BOUNDEDNESS OF COMMUTATORS OF FRACTIONAL HARDY OPERATORS WITH BESOV-LIPSCHITZ FUNCTIONS

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Abstract. In this paper, we establish two weighted integral inequalities for commutators of fractional Hardy operators with Besov-Lipschitz functions. The main result is that this kind of commutator, denoted by H_b^α, is bounded from $L^p_{x^\gamma}(\mathbf{R}_+)$ to $L^q_{x^\delta}(\mathbf{R}_+)$ with the bound explicitly worked out.

Key words: *fractional Hardy operator, commutator, Besov-Lipschitz function*

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1 Introduction and Main Results

Let f be a non-negative integrable function on $\mathbf{R}_+ = (0,\infty)$. The classical Hardy operator and its adjoint operator are defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \qquad x>0$$

and

$$H^*f(x) := \int_x^\infty \frac{f(t)}{t}\mathrm{d}t, \qquad x>0.$$

The following well-known integral inequalities is due to Hardy (cf.[5,6]).

Theorem A. *If f is a non-negative measurable function on $\mathbf{R}_+$ and $1<p<\infty$, then the following two inequalities*

$$\|Hf\|_{L^p(\mathbf{R}_+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbf{R}_+)}$$

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and

$$\|H^* f\|_{L^p(\mathbf{R}_+)} \leq p\|f\|_{L^p(\mathbf{R}_+)}$$

hold, where the constants $\frac{p}{p-1}$ *and p are sharp.*

For the n-dimensional case, Lu[9] discussed the following Hardy operator defined on the product space,

$$\mathcal{H}f(x) := \frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n, \quad x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}_+^n \tag{1}$$

and the adjoint operator of the Hardy operator defined by

$$\mathcal{H}^* f(x) := \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\frac{f(t_1,\cdots,t_n)}{t_1\cdots t_n}\mathrm{d}t_1\cdots\mathrm{d}t_n, \quad x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}_+^n, \tag{2}$$

where $\mathbf{R}_+^n=(0,\infty)^n$ and f is a non-negative measurable function on $\mathbf{R}_+^n$.

In [9], the following Theorem B is obtained.

Theorem B. *Suppose that f is any non-negative measurable function on* $\mathbf{R}_+^n$ *and* $1<p\leq q<\infty$. *Then the Hardy operator* $\mathcal{H}$ *defined by* (1) *is bounded from* $L^p(\mathbf{R}_+^n, x^\gamma)$ *to* $L^q(\mathbf{R}_+^n, x^\delta)$, *that is, the inequality*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}} \tag{3}$$

holds for some constant C, if and only if

$$\gamma<\mathbf{p}-\mathbf{1} \quad \text{and} \quad \delta=\frac{q}{p}(\gamma+\mathbf{1})-\mathbf{1}. \tag{4}$$

Moreover, if the conditions in (4) *are satisfied, then we have*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}f(x))^q x^\beta \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\prod_{i=1}^n \frac{q}{r(q-\delta_i-1)}\right)^{\frac{1}{r}}\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}}; \tag{5}$$

and the adjoint operator of the Hardy operator $\mathcal{H}^*$ *defined by* (2) *is also bounded from* $L^p(\mathbf{R}_+^n, x^\gamma)$ *to* $L^q(\mathbf{R}_+^n, x^\delta)$, *that is, the inequality*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}^*f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}} \tag{6}$$

holds for some constant C, if and only if

$$\gamma+\mathbf{1}>\mathbf{0} \quad \text{and} \quad \delta=\frac{q}{p}(\gamma+\mathbf{1})-\mathbf{1}. \tag{7}$$

Furthermore, if the conditions in (7) *are satisfied, then we have*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}^*f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\prod_{i=1}^n \frac{q}{r(\delta_i+1)}\right)^{\frac{1}{r}}\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}}, \tag{8}$$