Anal. Theory Appl.
Vol. 28, No. 1 (2012), 95–100

# A MATHEMATICAL PROOF OF A PROBABILISTIC MODEL OF HARDY'S INEQUALITY

Prateek K

(*Taleigao Plateau Goa University, India*)

Received June 23, 2010; Revised Feb. 14, 2012

**Abstract.** In this paper using an argument from [1] , we prove one of the probabilistic version of Hardy's inequality.

**Key words:** *random variables, uniform partition, Hardy's inequality*

**AMS (2010) subject classification:** 11K99, 11P55

## 1 Introduction

Hardy's inequality is defined as for a constant $c > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1$$

for all functions $f \in L^1([0,2\pi))$ with $\hat{f}(n) = 0$ for $n < 0$. This inequality is not true for all functions $f \in L^1([0,2\pi))$, which can be seen by letting $f$ to be the Fejér kernel of order $N$ for large enough $N$.

When McGehee, Pigno and Smith[3] proved the Littlewood conjecture, many questions were asked of how Hardy's inequality can be generalized for all functions $f \in L^1([0,2\pi))$. For instance, one of the expected generalizations is the following:

$$\sum_{n>0} \frac{\hat{f}(n)|}{n} \leq c\|f\|_1 + c\sum_{n>0} \frac{|\hat{f}(-n)|}{n}, \qquad f \in L^1([0,2\pi)),$$

where $c > 0$ is an absolute constant.

In this paper, we prove one version of Hardy's inequality for functions whose Fourier coefficients $\hat{f}(n)$ are random variables on $(0,1)$ for $n > 0$ without conditions on $\hat{f}(n)$ for $n < 0$.

In my proof use a technique that was motivated by Körner[1], who used this technique in a different problem to modify a result of Byrnes (see [1]).

In the sequel, $[0,2\pi)$ denotes the unit circle, $L^1([0,2\pi)$ the space of integrable functions on $[0,2\pi), \mu$ the Lebesgue measure, and $B_j$ the set of integers in the interval $[4^{j-1},4^j)$.

## 2 Basic Lemmas

In this section, I am going to prove some basic lemmas required for our purpose.

**Lemma 2.1.** *Let $X_1, X_2, \cdots, X_N$ be independent random variables such that*

$$|X_j| \leq 1 \qquad for\ each \qquad j, 1 \leq j \leq N,$$

*and write*

$$S_N = X_1 + X_2 + \cdots + X_N.$$

*Then, for any $\lambda > 0$,*

$$Pr(|S_N - ES_N| \geq \lambda) \leq 4\exp(-\frac{\lambda^2}{100N}).$$

For the proof, see [4, p.398].

The idea of the following proof is due to Köner[1]. The statement of the lemma was observed by Kahane [2] without proof.

**Lemma 2.2.** *Let $(r_k)$ be a sequence of independent, zero mean random variables defined on the interval $(0,1)$with $|r_k| \leq 1$ for all k. Let*

$$f_n(\theta,t) = \sum_{p=1}^{n} r_p(t)e^{ip\theta} \quad for \quad t \in (0,1) \quad and \quad \theta \in [0,2\pi).$$

*Then for $n \geq 27$ and $\lambda \geq 2 \times 2$,*

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \geq \lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{400}}.$$

*Proof.* By applying Lemma 2.1, we find that for fixed $\theta \in [0,2\pi)$,

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \geq \lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{100}}.$$

Let $(\theta_k)_{k=1}^{n^2}$ be a uniform partition of the unit circle. For fixed $t \in (0,1)$ and $\theta_k \in [0,2\pi)$ and for all $\theta$ with $|\theta - \theta_k| \leq 2\pi/n^2$, we have

$$|f_n(\theta,t) - f_n(\theta_k,t)| \leq \sum_{p=1}^{n} |r_p(t)||e^{ip\theta} - e^{ip\theta_k}| \leq 2\sum_{p=1}^{n} \frac{2\pi}{n^2}p = \frac{2\pi(n+1)}{n}.$$