

## ON THE GENERALIZED RESOLVENT OF LINEAR PENCILS IN BANACH SPACES

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**Abstract.** Utilizing the stability characterizations of generalized inverses of linear operator, we investigate the existence of generalized resolvent of linear pencils in Banach spaces. Some practical criteria for the existence of generalized resolvents of the linear pencil  $\lambda \rightarrow T - \lambda S$  are provided and an explicit expression of the generalized resolvent is also given. As applications, the characterization for the Moore-Penrose inverse of the linear pencil to be its generalized resolvent and the existence of the generalized resolvents of linear pencils of finite rank operators, Fredholm operators and semi-Fredholm operators are also considered. The results obtained in this paper extend and improve many results in this area.

**Key words:** *generalized inverse, generalized resolvent, linear pencils, Moore-Penrose inverse, Fredholm operator, semi-Fredholm operator*

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### 1 Introduction and Preliminaries

Let  $X$  and  $Y$  be two Banach spaces. Let  $B(X, Y)$  denote the Banach space of all bounded linear operators from  $X$  into  $Y$ . We write  $B(X)$  as  $B(X, X)$ . The identity operator will be denoted by  $I$ . For any  $T \in B(X, Y)$ , we denote by  $N(T)$  and  $R(T)$  the null space and the range of  $T$ , respectively.

The resolvent set  $\rho(T)$  of  $T \in B(X)$  is, by definition, the set of all complex number  $\lambda \in \mathbf{C}$  such that  $T - \lambda I$  is invertible in  $B(X)$ . And its resolvent  $R(T, \lambda) = (T - \lambda I)^{-1}$  is an analytic

function on  $\rho(T)$  since it satisfies the resolvent identity

$$R(T, \lambda) - R(T, \mu) = (\lambda - \mu)R(T, \lambda)R(\mu), \quad \forall \lambda, \mu \in \rho(T).$$

The spectrum  $\sigma(T)$  is the complement of  $\rho(T)$  in  $\mathbb{C}$ . As we all know, the spectral theory plays a fundamental role in functional analysis. If the operator  $T - \lambda I$  has a generalized inverse, we can consider the generalized resolvent and generalized spectrum. Some properties of the classical spectrum  $\sigma(T)$  remain true in the case of the generalized one<sup>[1-4]</sup>. Recall that an operator  $S \in B(Y, X)$  is said to be an inner inverse of  $T \in B(X, Y)$  if  $TST = T$  and an outer inverse if  $STS = S$ . If  $S$  is both an inner inverse and outer inverse of  $T$ , then  $S$  is called a generalized inverse of  $T$ <sup>[5]</sup>. We always write the generalized inverse of  $T$  by  $T^+$ . If  $T$  has a bounded generalized inverse  $T^+$ , then  $TT^+$  and  $T^+T$  are projectors with

$$\begin{aligned} R(TT^+) &= R(T), & R(T^+T) &= R(T^+), & N(T^+T) &= N(T), & N(TT^+) &= N(T^+) \\ X &= N(T) \oplus R(T^+), & Y &= N(T^+) \oplus R(T). \end{aligned}$$

Let us recall the concept of generalized resolvent. Let  $T \in B(X)$  and  $U$  be an open set in the complex plane. The function

$$U \ni \lambda \rightarrow R_g(T, \lambda) \in B(X)$$

is said to be a generalized resolvent of  $T - \lambda I$  on  $U$  if

- (1)  $(T - \lambda I)R_g(T, \lambda)(T - \lambda I) = T - \lambda I$  for all  $\lambda \in U$ ;
- (2)  $R_g(T, \lambda)(T - \lambda I)R_g(T, \lambda) = R_g(T, \lambda)$  for all  $\lambda \in U$ ;
- (3)  $R_g(T, \lambda) - R_g(T, \mu) = (\lambda - \mu)R_g(T, \lambda)R_g(T, \mu)$  for all  $\lambda$  and  $\mu$  in  $U$ .

The first two conditions say that  $R_g(T, \lambda)$  is a generalized inverse of  $T - \lambda I$  for each  $\lambda \in U$ , while the third one is an analogue of the classical resolvent identity. We also refer to it as the generalized resolvent identity, which plays an important role in the spectrum since it assures that  $R_g(T, \lambda)$  is locally analytic. The generalized resolvents has been widely used in many fields such as spectrum theory and theory of Fredholm operators<sup>[1-4, 6]</sup>. According to M. A. Shubin<sup>[7]</sup>, there exists a continuous generalized inverse function (satisfying (1) and (2) but not possibly (3)) meromorphic in the Fredholm domain  $\rho_\phi(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Fredholm}\}$ . And he points out that it remains an open problem whether or not this can be done while also satisfying (3), i.e., it is not known whether generalized resolvents always exist. Many authors have been interested in the existence problem for the generalized resolvents of linear operators in [1-4,6]. In [2], C. Badea and M. Mbekhta proved that  $T - \lambda I$  has an analytic generalized resolvent in a neighborhood of 0 if and only if  $T$  has a generalized inverse and  $N(T) \subset R(T^m), \forall m \in \mathbb{N}$ . It is worth mentioning that the condition  $N(T) \subset R(T^m)$  is not easy to be verified and its geometric significance is vague. In [3], C. Badea and M. Mbekhta introduced the concept of linear pencil and its generalized resolvents.