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L^p INEQUALITIES AND ADMISSIBLE OPERATOR FOR POLYNOMIALS

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Abstract. Let $p(z)$ be a polynomial of degree at most n. In this paper we obtain some new results about the dependence of

$$\left\| p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right\|_s$$

on $\|p(z)\|_s$ for every $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \le 1,\ |\beta| \le 1,\ R > r \geqslant 1$, and $s > 0$. Our results not only generalize some well known inequalities, but also are variety of interesting results deduced from them by a fairly uniform procedure.

Key words: *L^p inequality polynomials, Rouche's theorem, admissible operator*

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1 Introduction and Statement of Results

Let P_n be the class of all complex polynomials

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

of degree at most n and $p'(z)$ its derivative. For $p \in P_n$, define

$$\|p(z)\|_s := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^s \right\}^{\frac{1}{s}}, \qquad 1 \le s < \infty$$

and

$$\|p(z)\|_\infty := \max_{|z|=1} |p(z)|.$$

According to a famous result Known as Bernstein's inequality[4], we have

$$\|p'(z)\|_\infty \leq n\|p(z)\|_\infty. \tag{1}$$

Also concerning the maximum modulus of $p(z)$ on $|z| = R > 1$, we have

$$\|p(Rz)\|_\infty \leq R^n\|p(z)\|_\infty \tag{2}$$

(for reference see [11]). Zygmund[13] has shown

$$\|p'(z)\|_s \leq n\|p(z)\|_s, \qquad s \geqslant 1. \tag{3}$$

whereas we can deduce the following inequality by applying a result of Hardy [9],

$$\|p(Rz)\|_s \leq R^n\|p(z)\|_s, \quad R > 1, \quad s > 0. \tag{4}$$

Also Arestov[1] proved that (3) remains true for $0 < s < 1$ as well. It is clear that the inequalities (1) and (2) can be obtained by letting $s \longrightarrow \infty$ in the inequalities (3) and (4) respectively. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, the inequalities (3) and (4) can be improved. In fact, it was shown by De-Bruijn[6] for $s \geqslant 1$ and Rahman and Schmeisser[12] extended it for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, the inequality (3) can be replaced by

$$\|p'(z)\|_s \leq n\frac{\|p(z)\|_s}{\|1+z\|_s}, \qquad s > 0. \tag{5}$$

Also Boas and Rahman[5] proved for $s \geqslant 1$ and Rahman and Schmeisser[12] extended it for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, the inequality (4) can be replaced by

$$\|p(Rz)\|_s \leq \frac{\|R^n z + 1\|_s}{\|1+z\|_s}\|p(z)\|_s, \quad R > 1, \quad s > 0. \tag{6}$$

Aziz and Rather[2] obtained a generalization of the inequalities (3) and (4). In fact, they have shown that if $p \in P_n$, then for every $R > 1$ and $s \geqslant 1$,

$$\|p(Rz) - p(z)\|_s \leq (R^n - 1)\|p(z)\|_s. \tag{7}$$

Recently Aziz and Rather [3] considered a more general problem of investigating the dependence of

$$\|p(Rz) - \beta p(rz)\|_s \quad on \quad \|p(z)\|_s$$

for every $\beta \in \mathbf{C}$ with $|\beta| \leq 1,\ R > r \geqslant 1,\ s > 0$ and extended the inequality (7) for $0 < s < 1$ as following.