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BMO ESTIMATES FOR MULTILINEAR FRACTIONAL INTEGRALS*

Xiangxing Tao and Yunpin Wu
(*Zhejiang University of Science and Technology, China)*

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Abstract. In this paper, the authors prove that the multilinear fractional integral operator $T_{\Omega,\alpha}^{A_1,A_2}$ and the relevant maximal operator $M_{\Omega,\alpha}^{A_1,A_2}$ with rough kernel are both bounded from $L^p(1<p<\infty)$ to L^q and from L^p to $L^{n/(n-\alpha),\infty}$ with power weight, respectively, where

$$T_{\Omega,\alpha}^{A_1,A_2}(f)(x)=\int_{\mathbf{R}^n}\frac{R_{m_1}(A_1;x,y)R_{m_2}(A_2;x,y)}{|x-y|^{n-\alpha+m_1+m_2-2}}\Omega(x-y)f(y)\mathrm{d}y$$

and

$$M_{\Omega,\alpha}^{A_1,A_2}(f)(x)=\sup_{r>0}\frac{1}{r^{n-\alpha+m_1+m_2-2}}\int_{|x-y|<r}\prod_{i=1}^{2}R_{m_i}(A_i;x,y)\Omega(x-y)f(y)|\mathrm{d}y,$$

and $0<\alpha<n$, $\Omega\in L^s(S^{n-1})(s\geq 1)$ is a homogeneous function of degree zero in $\mathbf{R}^n$, A_i is a function defined on $\mathbf{R}^n$ and $R_{m_i}(A_i;x,y)$ denotes the m_i-th remainder of Taylor series of A_i at x about y. More precisely, $R_{m_i}(A_i;x,y)=A_i(x)-\sum_{|\gamma|<m_i}\frac{1}{\gamma!}D^\gamma A_i(y)(x-y)^r$, where $D^\gamma(A_i)\in \mathrm{BMO}(\mathbf{R}^n)$ for $|\gamma|=m_i-1(m_i>1)$, $i=1,2$.

Key words: *multilinear operator, fractional integral, rough kernel,* BMO

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1 Introduction

As two of the most important operators in harmonic analysis, the fractional integral operator $T_{\Omega,\alpha}$ and the corresponding maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x):=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\mathrm{d}y, \tag{1.1}$$

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$$M_{\Omega,\alpha}f(x) := \sup_{h>0}\int_{|x-y|<h}|\Omega(x-y)f(y)|\mathrm{d}y, \tag{1.2}$$

where $0<\alpha<n$, $1/q=1/p-\alpha/n$ and $\Omega\in L^s(S^{n-1})(s\ge n/(n-\alpha))$ is homogeneous of degree zero in $\mathbf{R}^n$. In 1993 and 1998, Chanillo [1] and Ding [7] proved that $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are bounded from $L^p(1<p<\infty)$ to L^q respectively. In 1997, Ding [2] gave that if $-1<\beta<0$ and $f\in L^1(|x|^{\beta(n-\alpha)/n})$, then $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are both bounded from $L^1(|x|^{\beta(n-\alpha)/n})$ to $L^{n/(n-\alpha),\infty}$.

It is well known that the study of multilinear fractional integral operators are received increasing attentions. Let $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$, and $\gamma_i(i=1,2,\cdots,n)$ be nonnegtive integers. Denote $|\gamma|=\sum_{i=1}^{n}\gamma_i$, $\gamma!=\gamma_1!\gamma_2!\cdots\gamma_n!$, $x^\gamma=x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n}$

$$D^\gamma=\frac{\partial^{|\gamma|}}{\partial^{\gamma_1}x_1\partial^{\gamma_2}x_2\cdots\partial^{\gamma_n}x_n}.$$

Suppose that A is a function defined on $\mathbf{R}^n$. Denote by $R_m(A;x,y)$ the m-th order remainder of the Taylor series of A at x about y, that is, $R_m(A;x,y)=A(x)-\sum_{|\gamma|<m}\frac{1}{\gamma!}D^\gamma A(y)(x-y)^r$, $m\ge1$. Then the multilinear fractional integral operator $T^A_{\Omega,\alpha}$ is defined by

$$T^A_{\Omega,\alpha}f(x) := \int_{\mathbf{R}^n}\frac{\Omega(x-y)R_m(A;x,y)}{|x-y|^{n-\alpha+m-1}}f(y)\mathrm{d}y \tag{1.3}$$

and the relevant maximal operator $M^A_{\Omega,\alpha}$ is given by

$$M^A_{\Omega,\alpha}f(x) := \sup_{r>0}\frac{1}{r^{n-\alpha+m-1}}\int_{|x-y|<r}|\Omega(x-y)R_m(A;x,y)f(y)|\mathrm{d}y. \tag{1.4}$$

In 2001, Ding[3] proved that if $D^\gamma A\in L^r(\mathbf{R}^n)(1<r\le\infty,|\gamma|=m-1)$, then $T^A_{\Omega,\alpha}$, $M^A_{\Omega,\alpha}$ are both weighted bounded operators from $L^p(w^p)$ to $L^q(w^q)$ with the weight $w\in A(p,q)$ and from $L^p(1\le p<n/\alpha)$ to $L^{n/(n-\alpha),\infty}$ with the power weight. Obviously, when $m=1$, $T^A_{\Omega,\alpha}$ reduces to the commutator generated by the fractional integral $T_{\Omega,\alpha}$ and the function A. In 2002, Yang and Wu [9] proved that if $D^\gamma A\in \mathrm{BMO}(\mathbf{R}^n)$, then $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are bounded from $L^p(1<p<\infty)$ to L^q. In 2003, Lu and Zhang[5] proved that if $D^\gamma A\in\dot{\Lambda}_\beta$, $s>\frac{n}{n-(\alpha+2\beta)}$, $0<\beta<1$, $1/q=1/p-(\alpha+\beta)/n$, then $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are bounded from $L^p(1<p<\frac{n}{\alpha+\beta})$ to L^q and from L^1 to $L^{n/n-\alpha-\beta,\infty}$. In 2001, Lu and Ding[4] showed that if $D^\gamma A_j\in\mathrm{BMO}(\mathbf{R}^n)$, than the operator

$$T^{A_1,A_2,\cdots,A_k}_{\Omega,\alpha}f(x) := \int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}}\prod_{j=1}^{k}R_{m_j}(A_j;x,y)f(y)\mathrm{d}y \tag{1.5}$$

with $N=\sum_{j=1}^{k}(m_j-1)(m_j\ge2)$ and the relevant maximal operator

$$M^{A_1,A_2,\cdots,A_k}_{\Omega,\alpha}f(x) := \sup_{r>0}\frac{1}{r^{n-\alpha+N}}\int_{|x-y|<r}|\Omega(x-y)\prod_{j=1}^{k}R_{m_j}(A_j;x,y)f(y)|\mathrm{d}y \tag{1.6}$$