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STANCU POLYNOMIALS BASED ON THE Q-INTEGERS

Xueyan Xiang
(*Lishui University, China*)

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Abstract. A new generalization of Stancu polynomials based on the q-integers and a nonnegative integer s is firstly introduced in this paper. Moreover, the shape-preserving and convergence properties of these polynomials are also investigated.

Key words: *Stancu polynomial, q-integer, q-derivative, shape-preserving property, convergence rate, modulus of continuity*
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1 Introduction

In 1981 Stancu proposed a kind of generalized Bernstein polynomials, namely Stancu polynomials, which was defined as:

Definition 1[1]. Let s be an integer and $0 \le s < \frac{n}{2}$, for $f \in C[0,1]$,

$$L_{n,s}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k,s}(x), \tag{1.1}$$

where

$$b_{n,k,s}(x) = \begin{cases} (1-x)p_{n-s,k}(x), & 0 \le k < s, \\ (1-x)p_{n-s,k}(x) + xp_{n-s,k-s}(x), & s \le k \le n-s, \\ xp_{n-s,k-s}(x), & n-s < k \le n, \end{cases}$$

and $p_{j,k}(x)$ are the base functions of Bernstein polynomials.

It is not difficult to see that for $s = 0,\ 1$ the Stancu polynomials are just the classical Bernstein polynomials. For $s \ge 2$, these polynomials possess many remarkable properties, which have made them an area of intensive research (see [2, 3, 4, 5]).

Throughout this paper we employ the following notations of q-Calculus. Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] = \begin{cases} \dfrac{1-q^k}{1-q}, & q \ne 1 \\ k, & q = 1, \end{cases}$$

$$[k]! = \begin{cases} [k][k-1]\cdots[1], & k \geq 1 \\ 1, & k = 0. \end{cases}$$

For $n,\ k,\ n \geq k \geq 0$, q-binomial coefficients are defined naturally as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Now let's introduce a new generalization of Stancu polynomials as below.

Definition 2. Let s be an integer and $0 \leq s < \frac{n}{2}$, $q > 0,\ n > 0$, for $f \in C[0,1]$,

$$L_{n,s}(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k,s}(q;x), \tag{1.2}$$

where

$$b_{n,k,s}(q;x) = \begin{cases} (1-q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leq k < s, \\ (1-q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s \leq k \leq n-s, \\ q^{n-k}xp_{n-s,k-s}(q;x), & n-s < k \leq n, \end{cases}$$

and

$$p_{n-s,k}(q;x) = \begin{bmatrix} n-s \\ k \end{bmatrix} x^k \prod_{l=0}^{n-s-k-1}(1-q^l x), \qquad k = 0,\ 1,\ \cdots,\ n-s.$$

(agree on $\prod_{l=0}^{0} = 1$).

It is worth mentioning that the q-Stancu polynomials defined as (1.2) differ essentially from the q-Stancu polynomials in [6]. To get their q-Stancu polynomials in [6] the authors just generalized the control points of the Stancu polynomials based on the q-integers leaving alone the basis functions. While in our q-Stancu polynomials both the control points and the basis functions are the q-analogue of those in Stancu polynomials. As a result, it is not a strange thing that these two q-Stancu polynomials behave quite differently properties, especially in the approximation problem.

It can be easily verified that in case $q = 1$, $L_{n,s}(f,q;x)$ reduce to the Stancu polynomials and in case $s = 0,\ 1$, $L_{n,s}(f,q;x)$ coincide with the q-Bernstein polynomials which are defined by Phillips in [7] and have been intensively investigated during these years (see [8-12]).

By some direct calculations, one can get the following two representations: for $f \in C[0,1]$, an integer s and $0 \leq s < \frac{n}{2}$,

$$L_{n,s}(f,q;x) = \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)f\left(\frac{[k]}{[n]}\right) + q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right)\right\}p_{n-s,k}(q;x); \tag{1.3}$$