STANCU POLYNOMIALS BASED ON THE Q-INTEGERs

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Abstract. A new generalization of Stancu polynomials based on the q-integers and a nonnegative integer \( s \) is firstly introduced in this paper. Moreover, the shape-preserving and convergence properties of these polynomials are also investigated.

Key words: Stancu polynomial, q-integer, q-derivative, shape-preserving property, convergence rate, modulus of continuity

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1 Introduction

In 1981 Stancu proposed a kind of generalized Bernstein polynomials, namely Stancu polynomials, which was defined as:

\[ L_{n,s}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k,s}(x), \quad (1.1) \]

where

\[ b_{n,k,s}(x) = \begin{cases} (1-x)p_{n-s,k}(x), & 0 \leq k < s, \\ (1-x)p_{n-s,k}(x) + xp_{n-s,k-s}(x), & s \leq k \leq n-s, \\ xp_{n-s,k-s}(x), & n-s < k \leq n, \end{cases} \]

and \( p_{j,k}(x) \) are the base functions of Bernstein polynomials.

It is not difficult to see that for \( s = 0, 1 \) the Stancu polynomials are just the classical Bernstein polynomials. For \( s \geq 2 \), these polynomials possess many remarkable properties, which have made them an area of intensive research (see [2, 3, 4, 5]).

Throughout this paper we employ the following notations of \( q \)-Calculus. Let \( q > 0 \). For each nonnegative integer \( k \), the \( q \)-integer \([k]\) and the \( q \)-factorial \([k]!\) are defined by

\[ [k] = \begin{cases} 1-q^k, & q \neq 1 \\ 1, & q = 1, \end{cases} \]

\[ [k]! = \begin{cases} 1-q^{k+1}, & q \neq 1 \\ 1, & q = 1, \end{cases} \]
For \( n, q \geq n \geq 0 \), \( q \)-binomial coefficients are defined naturally as

\[
[k]! = \begin{cases} 
[k][k-1]\cdots[1], & k \geq 1 \\
1, & k = 0.
\end{cases}
\]

Now let’s introduce a new generalization of Stancu polynomials as below.

**Definition 2.** Let \( s \) be an integer and \( 0 \leq s < \frac{n}{2} \), \( q > 0 \), \( n > 0 \), for \( f \in C[0,1] \),

\[
L_{n,s}(f,q;x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) b_{n,k,s}(q;x), \quad (1.2)
\]

where

\[
b_{n,k,s}(q;x) = \begin{cases} 
(1 - q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leq k < s, \\
(1 - q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s \leq k \leq n-s, \\
q^{n-k}xp_{n-s,k-s}(q;x), & n-s < k \leq n,
\end{cases}
\]

and

\[
p_{n-s,k}(q;x) = \begin{bmatrix} n-s \\ k \end{bmatrix} x^k \prod_{l=0}^{n-s-k-1} (1 - q^l x), \quad k = 0, 1, \ldots, n-s.
\]

(agree on \( \prod_{l=0}^{0} = 1 \)).

It is worth mentioning that the \( q \)-Stancu polynomials defined as (1.2) differ essentially from the \( q \)-Stancu polynomials in [6]. To get their \( q \)-Stancu polynomials in [6] the authors just generalized the control points of the Stancu polynomials based on the \( q \)-integers leaving alone the basis functions. While in our \( q \)-Stancu polynomials both the control points and the basis functions are the \( q \)-analogue of those in Stancu polynomials. As a result, it is not a strange thing that these two \( q \)-Stancu polynomials behave quite differently properties, especially in the approximation problem.

It can be easily verified that in case \( q = 1 \), \( L_{n,s}(f,q;x) \) reduce to the Stancu polynomials and in case \( s = 0, 1 \), \( L_{n,s}(f,q;x) \) coincide with the \( q \)-Bernstein polynomials which are defined by Phillips in [7] and have been intensively investigated during these years (see [8-12]).

By some direct calculations, one can get the following two representations: for \( f \in C[0,1] \), an integers and \( 0 \leq s < \frac{n}{2} \),

\[
L_{n,s}(f,q;x) = \sum_{k=0}^{n-s} \left\{ (1 - q^{n-k-s}x) f \left( \frac{k}{n} \right) + q^{n-k-s}x f \left( \frac{k+s}{n} \right) \right\} p_{n-s,k}(q;x); \quad (1.3)
\]