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GENERALIZATIONS OF THE SUZUKI AND KANNAN FIXED POINT THEOREMS IN G-CONE METRIC SPACES

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Abstract. In this paper we develop the Banach contraction principle and Kannan fixed point theorem on generalized cone metric spaces. We prove a version of Suzuki and Kannan type generalizations of fixed point theorems in generalized cone metric spaces.

Key words: *fixed point, D-metric space, 2-metric space, generalized cone metric space, Kannan mapping, generalized Kannan mapping, contractive mapping*
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1 Introduction

Recently, non-convex analysis has found some applications in optimization theory, and so there have been some investigations about non-convex analysis, especially ordered normed spaces, normal cones and topical functions. Huang and Zhang[5] have replaced the real numbers by an ordering Banach space and define cone metric spaces. They have proved some fixed point theorems of contractive mappings on cone metric spaces.

Let $(E,\|.\|)$ be a real Banach space and P be a subset of E, then P is called a cone whenever

(1) P is closed, non-empty and $P\neq\{0\}$.

(2) $ax+by\in P$ for all $x,y\in P$ and non-negative real numbers a,b.

(3) $P\cap(-P)=\{0\}$.

For a given cone $P\subseteq E$, we can define a partial ordering $\preceq$ with respect to P by $x\preceq y$ if and only if $y-x\in P$. We write $x\prec y$ if $x\preceq y$ and $x\neq y$ and write $x\ll y$ if $y-x\in$ int P, where int$P$ denotes the interior of P. The cone P is normal if there is a number $M>0$ such that for all $x,y\in E$

$$0\preceq x\preceq y\Longrightarrow\|x\|\leq M\|y\|.$$

The least positive number M satisfying the above inequality is called the normal constant of P. Rezapour and Hamalbarani[8] proved that there are no normal cones with normal constant $M<1$ and for each $k>1$ there are cones with normal constant $M>k$. The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\in\mathbf{N}}$ is sequence in E such that $x_1\preceq x_2\preceq\cdots\preceq x_n\preceq y$ for some $y\in E$, then there is $x\in E$ such that $\lim_{n\to\infty}\|x_n-x\|=0$. Equivalently, the cone P is called regular if every decreasing sequence which is bounded from below is convergent.

A subset $F\subseteq E$ is called bounded if there exists $u\in E$ such that $x\preceq u$ for every $x\in F$, in this case we say u is an upper bound of F. Similarly the lower bounds can be defined for F. The set F is said to be bounded if F is bounded from both below and above. We say that E has the supremum property, if every non-empty above bounded subset of E has a least upper bound in E. It is easy to see that if E has the supremum property, then E is regular.

The concept of 2-Metric space as a generalization of metric space was first defined by Gahler[4] in 1963, but some other authors proved that there is no relation between these two concepts. Dhage[3] defined D-metric space as a generalization of the metric space, later Mustafa and Sims[7] showed that most of the claims concerning fundamental topological properties of D-metric spaces are incorrect and reintroduced a new structure of generalized metric spaces, called G-metric space. Recently Beg et, al. [2] introduced the G-cone metric space and proved some common fixed point theorems in this space.

Throughout this paper we suppose that $(E,\|.\|)$ is a real Banach space and P is a normal cone with a constant M and the supremum property in E and int$P\neq\emptyset$. The purpose of this paper is to generalize and unify a version of Banach contraction principle and Kannan fixed point theorem on complete generalized cone metric spaces. First we briefly recall the definitions and basic properties of generalized cone metric spaces. For more information we refer to the articles by Mustafa and Sims[7] and Beg et al[2].

Definition 1.1. Let X be a non-empty set, then the mapping $G: X\times X\times X\to E$ is called a generalized cone metric or simply a G-cone metric on X if it satisfies the following axioms:

(G_1) $G(x,y,z)=0$ if $x=y=z$.

(G_2) $G(x,x,y)\succ 0$, whenever $x\neq y$, for all $x,y\in X$.

(G_3) $G(x,x,y)\preceq G(x,y,z)$, whenever $y\neq z$.

(G_4) G is a symmetry function of its three variables.

(G_5) $G(x,y,z)\preceq G(x,y,u)+G(u,u,z)$ for all $x,y,z,u\in X$.