A CHARACTERIZATION FOR FRACTIONAL INTEGRALS
ON GENERALIZED MORREY SPACES

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Abstract. This paper concerns with the fractional integrals, which are also known as the
Riesz potentials. A characterization for the boundedness of the fractional integral operators
on generalized Morrey spaces will be presented. Our results can be viewed as a refinement
of Nakai’s[7].

Key words: fractional integrals, Morrey spaces

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1 Introduction

For $0 < \alpha < d$, we define the fractional integral (also known as the Riesz potential) $I_\alpha f$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \, dy, \quad x \in \mathbb{R}^d,$$

for any suitable function $f$ on $\mathbb{R}^d$. Clearly $I_\alpha f$ is well-defined for any locally bounded, compactly supported function $f$ on $\mathbb{R}^d$. It is well-known that $I_\alpha$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, that is,

$$\|I_\alpha f : L^q\| \leq C \|f : L^p\|$$

if and only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}.$$
with \(1 < p < \frac{d}{\alpha}\). This result was proved by Hardy and Littlewood\(^{[5,6]}\) and Sobolev\(^{[10]}\) around the 1930’s. Further development on the subject can be found in \([11, 12]\).

Next, let \(R^+ := (0, \infty)\). For \(1 \leq p < \infty\) and a suitable function \(\phi : R^+ \rightarrow R^+\), we define the generalized Morrey space \(L^{p, \phi} = L^{p, \phi}(R^d)\) to be the set of all functions \(f \in L^{p}_{\text{loc}}(R^d)\) for which

\[
\|f : L^{p, \phi}\| := \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p \, dy \right)^{1/p} < \infty.
\]

Here the supremum are taken over all open balls \(B = B(a, r)\) in \(R^d\) and \(\phi(B) = \phi(r)\), where \(r \in R^+\). For certain functions \(\phi\), the spaces \(L^{p, \phi}\) reduce to some classical spaces. For instance, if \(\phi(r) = r^{(\lambda - d)/p}\), where \(0 \leq \lambda \leq d\), then \(L^{p, \phi}\) is the classical Morrey space \(L^{p, \lambda}\). For a brief history of the Morrey space and related spaces, see \([8]\). For more recent results, see \([9, 13]\) and the references therein.

In this short paper, we shall revisit Nakai’s theorems on the fractional integrals on the generalized Morrey spaces\(^{[7]}\). In particular, we find that the sufficient condition imposed by Nakai for the boundedness of the operator turns out to be necessary. In other words, we obtain a characterization for which the fractional integral operators are bounded from \(L^{p, \phi}\) to \(L^{q, \psi}\).

## 2 Main Results

Let us begin with some assumptions and relevant facts that follow. As customary, the letters \(C, C_i, C_p\) and \(C_{p,q}\) denote positive constants, which may depend on the parameters such as \(\alpha\), \(p, q\) and the dimension \(d\) of the ambient space, but not on the function \(f\) or the variable \(x\). These constants may vary from line to line.

In the definition of \(L^{p, \phi}\), the function \(\phi\) is assumed to satisfy the following conditions:

\[
\phi \text{ is almost decreasing : } t \leq r \Rightarrow \phi(r) \leq C_1 \phi(t); \\
r^d \phi(r)^p \text{ is almost increasing : } t \leq r \Rightarrow r^d \phi(t)^p \leq C_2 r^d \phi(r)^p,
\]

with \(C_1, C_2 > 0\) being independent of \(r\) and \(t\). These two conditions imply that

\[
\phi \text{ satisfies the doubling condition : } 1 \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{C_3} \leq \frac{\phi(t)}{\phi(r)} \leq C_3,
\]

for some \(C_3 > 0\) (which is also independent of \(r\) and \(t\)). Throughout this paper, we shall always assume that \(\phi\) satisfies these conditions.